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ON CONFORMABLE LINEAR FRACTIONAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

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Abstract:

In this article, we introduce the concept of conformable linear fractional differential equations. General solutions for these equations are established. An attempt has been made to obtain particular integral for some special cases. This is achieved by obtaining a relationship between fractional integration and the usual integration. Finally, some real-world problems like the mass-spring-damper system and Gompertz Law of population growth are used to demonstrate the effectiveness of the theoretical findings.

Key Words: Conformable Fractional Derivative, Conformable Fractional Integral, Conformable Linear Fractional Differential, Mass-Spring-Damper System, Gompertz Law.

1. Introduction:

The discipline of the mathematical analysis dedicated to the study of operators having non-integer (arbitrary) order: fractional derivatives and integrals, is called Fractional Calculus (FC)[14][20][21]. FC has lured much attention in the last 30 years due to its robust and extensive use in many areas of science and engineering, in the field of modeling and control processes [7]. A search for the best-suited definition of fractional derivatives and integrals has led to a fair number of definitions in this field [10], some of the popular ones are given by Riemann-Liouville, Grunwald-Letnikov, Caputo, Weyl, [17][19-21][25], and more recently, Caputo and Fabrizio[8] and Atangana and Baleanu.[6]

One of the intricacies encountered by FC is what kind of fractional derivative will replace the ordinary derivative for a given problem. The reason for this hunt is devoted to the inconsistencies raised in real problems due to the lack of properties, such as the product, quotient, composition, and chain rule, and theorems like Rolle's and mean value in almost all fractional derivatives. To eradicate these difficulties, Khalil et al. [16] introduced a new limit-based definition of fractional derivatives known as a conformable fractional derivative. Researchers are drawn to this definition, as it resembles the standard derivative [1][15] and hence makes the computation much easier. Therefore, a generous number of works are carried out using this new definition [2-5,9,11-13,18,24]. Motivated by this new conformable derivative, we apply it to the well-accepted concept of ordinary linear differential equations with constant coefficients [23] and obtain the results for the corresponding fractional differential equations. Also, not to limit our findings just for the theoretical purpose, we implemented them to the mass-spring-damper system [13] and Gompertz Law [11].

The organization of this paper is as follows: In section 2, a relationship between fractional and usual integrals is established. Section 3 present the definition of conformable fractional linear differential equations and their general solutions, section 4 possesses particular integrals for these equations, and section 5 is devoted to the applicatory part. Finally, we conclude ourselves in section 6. Given below are some basic definitions.

Definition 1[16]:

Given a function $f : [0,\infty)\to \mathbb{R}$. Then, the conformable fractional derivative of order α is given by

$$T_\alpha(f)(t) = \lim_{\varepsilon\to 0}\frac{f\left(t+\varepsilon t^{1-\alpha}\right)-f(t)}{\varepsilon}$$

for all $t>0, \alpha\in(0,1]$."

Definition 2 [16]:

Let $\alpha\in(n,n+1]$ and f be an n-differentiable function at t, where $t>0$. Then, the conformable fractional derivative of f of order α is defined as

$$T_\alpha(f)(t) = \lim_{\varepsilon\to 0}\frac{f^{(\lceil\alpha\rceil-1)}\left(t+\varepsilon t^{(\lceil\alpha\rceil-\alpha)}\right)-f^{(\lceil\alpha\rceil-1)}(t)}{\varepsilon}$$

where $\lceil \alpha \rceil$ is the smallest integer greater than or equal to α ."

Definition 3 [16]:

The conformable fractional integral of order α is given by

$$I_\alpha(f)(t) = \int \frac{f(t)}{t^{1-\alpha}} dt,$$

where the integral is the usual Riemann improper integral and $\alpha \in (0, 1]$.

2. Relationship Between Fractional and Usual Integrals:

As we know, integration is a complex process, and to save ourselves from unnecessary calculations, in this section we tried to figure out a simple relationship between the fractional integral and the usual integral.

Theorem 2.1:

All the fractional integrals can be obtained from the formulae for usual integration by replacing s by $\frac{s^\alpha}{\alpha}$, $\alpha \in (0, 1]$.

Proof:

As we know, by definition of fractional integral

$$I_\alpha(f)(s) = \int \frac{f(s)}{s^{1-\alpha}} ds \qquad (1)$$

where I_α denotes the fractional integral of order α and $\int f(x)dx$ denotes the usual integration.

On replacing s by $\frac{s^\alpha}{\alpha}$, this equation becomes

$$I_\alpha(f)\left(\frac{s^\alpha}{\alpha}\right) = \int \frac{f\left(\frac{s^\alpha}{\alpha}\right)}{s^{1-\alpha}} ds \qquad (2)$$

Putting $\frac{s^\alpha}{\alpha} = x$ and differentiating both sides, we obtain

$$\alpha \frac{s^{\alpha-1}}{\alpha} ds = dx \Rightarrow s^{\alpha-1} ds = dx .$$

Using these, equation (2) becomes

$$I_\alpha(f)\left(\frac{s^\alpha}{\alpha}\right) = \int f(x) dx \cdot$$

3. Conformable Linear Fractional Differential Equations with Constant Coefficients:

Here, we define conformable linear fractional differential equations (LFDEs) with constant coefficients and find their general solution for some special cases along with examples for each case.

Definition 3.1:

A conformable linear fractional differential equation with constant coefficients is that in which the dependent variable and its fractional differential coefficients occur only in the first degree of fractional order, their multiplication with each other is not allowed and the coefficients are all constants.

Case I: When the auxiliary equation has distinct real roots.

Theorem 3.2:

Let $\frac{d^{n\alpha}y}{ds^{n\alpha}} + a_{n-1}\frac{d^{(n-1)\alpha}y}{ds^{(n-1)\alpha}} + \ldots + a_1\frac{d^\alpha y}{ds^\alpha} + a_0 y = 0 \qquad (1)$

If $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the real roots, then $y = l_1y_1 + l_2y_2 + \ldots + l_ny_n$ is the general solution where $y_i = e^{\lambda_i \frac{s^\alpha}{\alpha}}$

$$\Rightarrow y = l_1 e^{\lambda_1 \frac{s^\alpha}{\alpha}} + l_2 e^{\lambda_2 \frac{s^\alpha}{\alpha}} + \ldots + l_n e^{\lambda_n \frac{s^\alpha}{\alpha}} .$$

Proof:

Let $S^{\alpha}y = \frac{d^{\alpha}y}{ds^{\alpha}}$, then the auxiliary equation for (1) is given by

$$\left(S^{\alpha n} + a_{n-1}S^{\alpha(n-1)} + \ldots + a_0 I\right)y = 0 \quad (2)$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the real roots of (2), and then it can be written as

$$\left(S^{\alpha} - \lambda_1\right)\left(S^{\alpha} - \lambda_2\right)\ldots\left(S^{\alpha} - \lambda_n\right)y = 0 \quad (3)$$

Then, the solution of equation (3) will be the solution of any one of the equations

$$\left(S^{\alpha} - \lambda_1\right)y = 0, \left(S^{\alpha} - \lambda_2\right)y = 0, \ldots, \left(S^{\alpha} - \lambda_n\right)y = 0 \quad (4)$$

However, $\left(S^{\alpha} - \lambda_i\right)y = 0$ implies $S^{\alpha}y - \lambda_i y = 0$, which is an LFDE and can be solved using the method given in section 3. Here, I.F. $= e^{I_{\alpha}(-\lambda_i)} = e^{-\lambda_i \frac{s^{\alpha}}{\alpha}}$ and thus, the solution is

$$ye^{-\lambda_i \frac{s^{\alpha}}{\alpha}} = l_i \Rightarrow y = l_i e^{\lambda_i \frac{s^{\alpha}}{\alpha}}$$ where l_i is a real constant.

Thus, the general solution can be written as

$$y = l_1 e^{\lambda_1 \frac{s^{\alpha}}{\alpha}} + l_2 e^{\lambda_2 \frac{s^{\alpha}}{\alpha}} + \ldots + l_n e^{\lambda_n \frac{s^{\alpha}}{\alpha}}$$, if all the roots are real and distinct.

Example 3.3:

Solving $\frac{d^{2\alpha}y}{ds^{2\alpha}} + 3\frac{d^{\alpha}y}{ds^{\alpha}} + 2y = 0$ yields $y = l_1 e^{-\frac{s^{\alpha}}{\alpha}} + l_2 e^{-2\frac{s^{\alpha}}{\alpha}}$, for some real constants l_1 and l_2.

Case II: When the auxiliary equation has repeated real roots.

Theorem 3.4:

Let λ_1 be a twice repeated root and $\lambda_3, \lambda_4, \ldots, \lambda_n$ are distinct real roots, then the general solution is obtained as

$$y = \left(l_1 \frac{s^{\alpha}}{\alpha} + l_2\right)e^{\lambda_1 \frac{s^{\alpha}}{\alpha}} + l_3 e^{\lambda_3 \frac{s^{\alpha}}{\alpha}} \ldots + l_n e^{\lambda_n \frac{s^{\alpha}}{\alpha}}.$$

Proof:

Since λ_1 is a twice repeated root, therefore we have

$$\left(S^{\alpha} - \lambda_1\right)^2 y = 0$$

$$\Rightarrow \left(S^{\alpha} - \lambda_1\right)\left(S^{\alpha} - \lambda_1\right)y = 0 \quad (1)$$

Let $\left(S^{\alpha} - \lambda_1\right)y = v$, then equation (1) becomes

$\Rightarrow \left(S^{\alpha} - \lambda_1\right)v = 0$ which is an LFDE and hence for some constant l_1, its solution is given by $v = l_1 e^{\lambda_1 \frac{s^{\alpha}}{\alpha}}$. Putting this value of v in $\left(S^{\alpha} - \lambda_1\right)y = v$, we obtain

$$\left(S^{\alpha} - \lambda_1\right)y = l_1 e^{\lambda_1 \frac{s^{\alpha}}{\alpha}}$$ which is again an LFDE.

Thus, I.F. $= e^{-\lambda_1 \frac{s^{\alpha}}{\alpha}}$ and hence, the solution is

$$y.e^{-\lambda_1 \frac{s^{\alpha}}{\alpha}} = I_{\alpha}\left(l_1 e^{\lambda_1 \frac{s^{\alpha}}{\alpha}}.e^{-\lambda_1 \frac{s^{\alpha}}{\alpha}}\right) + l_2$$

$$y.e^{-\lambda_1\frac{s^\alpha}{\alpha}} = I_\alpha\left(l_1\right)+l_2$$

$$\Rightarrow y.e^{-\lambda_1\frac{s^\alpha}{\alpha}} = l_1\frac{s^\alpha}{\alpha}+l_2$$

$$\Rightarrow y = \left(l_1\frac{s^\alpha}{\alpha}+l_2\right)e^{\lambda_1\frac{s^\alpha}{\alpha}}.$$

Thus, the general solution can be written as $y = \left(l_1\frac{s^\alpha}{\alpha}+l_2\right)e^{\lambda_1\frac{s^\alpha}{\alpha}}+l_3e^{\lambda_3\frac{s^\alpha}{\alpha}}...+l_ne^{\lambda_n\frac{s^\alpha}{\alpha}}$,where l_i 's are some real constants.

Similarly, for three repeated roots the general solution so obtained will be

$$y = \left(l_1\left(\frac{s^\alpha}{\alpha}\right)^2+l_2\frac{s^\alpha}{\alpha}+l_3\right)e^{\lambda_1\frac{s^\alpha}{\alpha}}+l_4e^{\lambda_4\frac{s^\alpha}{\alpha}}...+l_ne^{\lambda_n\frac{s^\alpha}{\alpha}}$$

Example 3.5:

The solution for $\frac{d^{2\alpha}y}{ds^{2\alpha}}+2\frac{d^\alpha y}{ds^\alpha}+y=0$ is given by $y=\left(l_1\frac{s^\alpha}{\alpha}+l_2\right)e^{-\frac{s^\alpha}{\alpha}}$ where l_1 and l_2 are some real constants.

Case III: When the roots are complex.

Theorem 3.6:

Let $\lambda_1 = a\pm ib$ be two complex roots and $\lambda_3,\lambda_4,...,\lambda_n$ are distinct real roots, then the general solution for this case is

$$y = e^{a\frac{s^\alpha}{\alpha}}\left(l_1\cos b\frac{s^\alpha}{\alpha}+l_2\sin b\frac{s^\alpha}{\alpha}\right)+l_3e^{\lambda_3\frac{s^\alpha}{\alpha}}...+l_ne^{\lambda_n\frac{s^\alpha}{\alpha}}$$,for some real constants denoted by l_i .

Proof:

Since $a\pm ib$ are roots of the auxiliary equation, thus

$$y = \beta e^{(a+ib)\frac{s^\alpha}{\alpha}}+\delta e^{(a-ib)\frac{s^\alpha}{\alpha}}$$

$$= e^{a\frac{s^\alpha}{\alpha}}\left[\beta e^{ib\frac{s^\alpha}{\alpha}}+\delta e^{-ib\frac{s^\alpha}{\alpha}}\right]$$

$$= e^{a\frac{s^\alpha}{\alpha}}\left[\beta\left(\cos b\frac{s^\alpha}{\alpha}+i\sin b\frac{s^\alpha}{\alpha}\right)+\delta\left(\cos b\frac{s^\alpha}{\alpha}-i\sin b\frac{s^\alpha}{\alpha}\right)\right]$$

$$= e^{a\frac{s^\alpha}{\alpha}}\left[(\beta+\delta)\cos b\frac{s^\alpha}{\alpha}+i(\beta-\delta)\sin b\frac{s^\alpha}{\alpha}\right]$$

$$= e^{a\frac{s^\alpha}{\alpha}}\left[l_1\cos b\frac{s^\alpha}{\alpha}+l_2\sin b\frac{s^\alpha}{\alpha}\right]$$ {On choosing $l_1=\beta+\delta, l_2=\beta-\delta$ }

The general solution thus obtained is $y = e^{a\frac{s^\alpha}{\alpha}}\left(l_1\cos b\frac{s^\alpha}{\alpha}+l_2\sin b\frac{s^\alpha}{\alpha}\right)+l_3e^{\lambda_3\frac{s^\alpha}{\alpha}}...+l_ne^{\lambda_n\frac{s^\alpha}{\alpha}}$

Example 3.7:

$y=\left(l_1\cos\frac{s^\alpha}{\alpha}+l_2\sin\frac{s^\alpha}{\alpha}\right)$ can be obtained as a solution for $\frac{d^{2\alpha}y}{ds^{2\alpha}}+y=0$ with usual constants and notations.

Remark 3.8:

If $a\pm ib$, $a\pm ib$ are the repeated complex roots and $\lambda_5,\lambda_6,...,\lambda_n$ are the distinct real roots, then general solution is given by

$$y=e^{a\frac{s^\alpha}{\alpha}}\left(\left(l_1+l_2\frac{s^\alpha}{\alpha}\right)\cos b\frac{s^\alpha}{\alpha}+\left(l_3+l_4\frac{s^\alpha}{\alpha}\right)\sin b\frac{s^\alpha}{\alpha}\right)+l_5e^{\lambda_5\frac{s^\alpha}{\alpha}}...+l_ne^{\lambda_n\frac{s^\alpha}{\alpha}}.$$

Example 3.9:

On solving $\frac{d^{4\alpha}y}{ds^{4\alpha}}+2\frac{d^{2\alpha}y}{ds^{2\alpha}}+y=0$, we get $y=\left(\left(l_1+l_2\frac{s^\alpha}{\alpha}\right)\cos\frac{s^\alpha}{\alpha}+\left(l_3+l_4\frac{s^\alpha}{\alpha}\right)\sin\frac{s^\alpha}{\alpha}\right)$ as a solution with usual notations and constants. The proof is left as an exercise for the readers. {**Hint:** Roots of the auxiliary eqn. are $\pm i,\pm i$ }

Case IV: When the roots are surd.

Theorem 3.10:

If $a\pm\sqrt{b}$ are the surd roots and $\lambda_3,\lambda_4,...,\lambda_n$ are distinct real roots, then the general solution is given by

$$y=e^{a\frac{s^\alpha}{\alpha}}\left(l_1\cosh\sqrt{b}\frac{s^\alpha}{\alpha}+l_2\sinh\sqrt{b}\frac{s^\alpha}{\alpha}\right)+l_3e^{\lambda_3\frac{s^\alpha}{\alpha}}...+l_ne^{\lambda_n\frac{s^\alpha}{\alpha}}.$$

Proof:

Since $a\pm\sqrt{b}$ are roots of the auxiliary equation, thus

$$y=\beta e^{\left(a+\sqrt{b}\right)\frac{s^\alpha}{\alpha}}+\delta e^{\left(a-\sqrt{b}\right)\frac{s^\alpha}{\alpha}}$$

$$=e^{a\frac{s^\alpha}{\alpha}}\left[\beta e^{\sqrt{b}\frac{s^\alpha}{\alpha}}+\delta e^{-\sqrt{b}\frac{s^\alpha}{\alpha}}\right]$$

$$=e^{a\frac{s^\alpha}{\alpha}}\left[\beta\left(\cosh\sqrt{b}\frac{s^\alpha}{\alpha}+\sinh\sqrt{b}\frac{s^\alpha}{\alpha}\right)+\delta\left(\cosh\sqrt{b}\frac{s^\alpha}{\alpha}-\sinh\sqrt{b}\frac{s^\alpha}{\alpha}\right)\right]$$

As we know, $e^\theta=\cosh\theta+\sinh\theta, e^{-\theta}=\cosh\theta-\sinh\theta$

$$=e^{a\frac{s^\alpha}{\alpha}}\left[(\beta+\delta)\cosh\sqrt{b}\frac{s^\alpha}{\alpha}+(\beta-\delta)\sinh\sqrt{b}\frac{s^\alpha}{\alpha}\right]$$ {On choosing $l_1=\beta+\delta, l_2=\beta-\delta$ }

$$=e^{a\frac{s^\alpha}{\alpha}}\left[l_1\cosh\sqrt{b}\frac{s^\alpha}{\alpha}+l_2\sinh\sqrt{b}\frac{s^\alpha}{\alpha}\right]$$

Example 3.11:

$y=e^{\frac{s^\alpha}{\alpha}}\left[l_1\cosh\sqrt{2}\frac{s^\alpha}{\alpha}+l_2\sinh\sqrt{2}\frac{s^\alpha}{\alpha}\right]$ is the solution for LFDE $\frac{d^{2\alpha}y}{ds^{2\alpha}}-2\frac{d^\alpha y}{ds^\alpha}-y=0$ with roots of the auxiliary equation as $S^\alpha=1+\sqrt{2},1-\sqrt{2}$.

Remark 3.12:

If $a\pm\sqrt{b}$, $a\pm\sqrt{b}$ are repeated surd roots and $\lambda_5,\lambda_6,\ldots,\lambda_n$ are distinct real roots, then the general solution is given by

$$y=e^{a\frac{s^{\alpha}}{\alpha}}\left(\left(l_1+l_2\frac{s^{\alpha}}{\alpha}\right)\cosh\sqrt{b}\frac{s^{\alpha}}{\alpha}+\left(l_3+l_4\frac{s^{\alpha}}{\alpha}\right)\sinh\sqrt{b}\frac{s^{\alpha}}{\alpha}\right)+l_5e^{\lambda_5\frac{s^{\alpha}}{\alpha}}\ldots+l_ne^{\lambda_n\frac{s^{\alpha}}{\alpha}}.$$

Example 3.13:

$\frac{d^{4\alpha}y}{ds^{4\alpha}}-4\frac{d^{3\alpha}y}{ds^{3\alpha}}+2\frac{d^{2\alpha}y}{ds^{2\alpha}}+4\frac{d^{\alpha}y}{ds^{\alpha}}+y=0$ with auxiliary equation

$\left(S^{\alpha}-\left(1+\sqrt{2}\right)\right)\left(S^{\alpha}-\left(1-\sqrt{2}\right)\right)\left(S^{\alpha}-\left(1+\sqrt{2}\right)\right)\left(S^{\alpha}-\left(1-\sqrt{2}\right)\right)=0$ so that

$S^{\alpha}=1\pm\sqrt{2},1\pm\sqrt{2}$ has the solution

$$y=e^{\frac{s^{\alpha}}{\alpha}}\left(\left(l_1+l_2\frac{s^{\alpha}}{\alpha}\right)\cosh\sqrt{2}\frac{s^{\alpha}}{\alpha}+\left(l_3+l_4\frac{s^{\alpha}}{\alpha}\right)\sinh\sqrt{2}\frac{s^{\alpha}}{\alpha}\right).$$

4. Methods to Determine Particular Integral (P.I.) of $f\left(S^{\alpha}\right)y=S$, where $S^{\alpha}=\frac{d^{\alpha}}{ds^{\alpha}}$ is the Conformable Fractional Derivative of Order $\alpha\in(0,1]$

Section 4 consists of a general method to find the particular integral and also some direct formulae to get the particular integral of some special type of functions along with examples to justify each result.

4.1. General Method of Getting Particular Integral:

Theorem 4.1.1:

If S is a function of s, then $\frac{1}{S^{\alpha}-a}S=e^{a\frac{s^{\alpha}}{\alpha}}\int Se^{-a\frac{s^{\alpha}}{\alpha}}ds$, where S^{α} is the "conformable fractional derivative" of order $\alpha\in(0,1]$.

Proof:

Let $\frac{1}{S^{\alpha}-a}S=y$ (1)

$\Rightarrow\left(S^{\alpha}-a\right)y=S$ which is a LFDE and its solution is given by

$y.(I.F)=\int S(I.F)ds$, where I.F. $=e^{I_{\alpha}(-a)}=e^{-a\frac{s^{\alpha}}{\alpha}}$

$\Rightarrow y.e^{-a\frac{s^{\alpha}}{\alpha}}=\int Se^{-a\frac{s^{\alpha}}{\alpha}}ds$

$\Rightarrow y=e^{a\frac{s^{\alpha}}{\alpha}}\int Se^{-a\frac{s^{\alpha}}{\alpha}}ds$.

$\Rightarrow\frac{1}{S^{\alpha}-a}S=e^{a\frac{s^{\alpha}}{\alpha}}\int Se^{-a\frac{s^{\alpha}}{\alpha}}ds$ {From (1)}

Example 4.1.2:

Solve $\left(S^{2\alpha}+4\right)y=\tan 2\frac{s^{\alpha}}{\alpha}$.

Solution:

Here, the auxiliary equation is given by $\left(S^{2\alpha}+4\right)=0$

$\Rightarrow\left(S^{\alpha}+2i\right)\left(S^{\alpha}-2i\right)=0$

$\Rightarrow S^{\alpha} = \pm 2i$.

Thus, the complementary function is given by

C.F. $= l_1 \cos 2\frac{s^{\alpha}}{\alpha} + l_2 \sin 2\frac{s^{\alpha}}{\alpha}$. Now, for obtaining the particular integral, we shall use the general method as stated above.

$$\text{P.I.} = \frac{1}{\left(S^{2\alpha}+4\right)} \tan 2\frac{s^{\alpha}}{\alpha} = \frac{1}{\left(S^{\alpha}+2i\right)\left(S^{\alpha}-2i\right)} \tan 2\frac{s^{\alpha}}{\alpha} = \frac{1}{4i}\left[\frac{1}{S^{\alpha}-2i} - \frac{1}{S^{\alpha}+2i}\right] \tan 2\frac{s^{\alpha}}{\alpha} \qquad (1)$$

Now, $\frac{1}{S^{\alpha}-2i} \tan 2\frac{s^{\alpha}}{\alpha} = e^{2i\frac{s^{\alpha}}{\alpha}} I_{\alpha}\left(e^{-2i\frac{s^{\alpha}}{\alpha}} \tan 2\frac{s^{\alpha}}{\alpha}\right)$

$$= e^{2i\frac{s^{\alpha}}{\alpha}} I_{\alpha}\left(\left(\cos 2\frac{s^{\alpha}}{\alpha} - i \sin 2\frac{s^{\alpha}}{\alpha}\right) \frac{\sin 2\frac{s^{\alpha}}{\alpha}}{\cos 2\frac{s^{\alpha}}{\alpha}}\right) = e^{2i\frac{s^{\alpha}}{\alpha}} I_{\alpha}\left(\sin 2\frac{s^{\alpha}}{\alpha} - i\frac{1-\cos^2 2\frac{s^{\alpha}}{\alpha}}{\cos 2\frac{s^{\alpha}}{\alpha}}\right)$$

$$= e^{2i\frac{s^{\alpha}}{\alpha}} I_{\alpha}\left(\sin 2\frac{s^{\alpha}}{\alpha} - i\left(\sec 2\frac{s^{\alpha}}{\alpha} - \cos 2\frac{s^{\alpha}}{\alpha}\right)\right) = e^{2i\frac{s^{\alpha}}{\alpha}}\left[-\frac{\cos 2\frac{s^{\alpha}}{\alpha}}{2} - \frac{i}{2}\log\tan\left(\frac{\pi}{4}+\frac{s^{\alpha}}{\alpha}\right) + i\frac{\sin 2\frac{s^{\alpha}}{\alpha}}{2}\right]$$

$$= -\frac{e^{2i\frac{s^{\alpha}}{\alpha}}}{2}\left[\cos 2\frac{s^{\alpha}}{\alpha} - i\sin 2\frac{s^{\alpha}}{\alpha} + i\log\tan\left(\frac{\pi}{4}+\frac{s^{\alpha}}{\alpha}\right)\right] = -\frac{e^{2i\frac{s^{\alpha}}{\alpha}}}{2}\left[e^{-2i\frac{s^{\alpha}}{\alpha}} + i\log\tan\left(\frac{\pi}{4}+\frac{s^{\alpha}}{\alpha}\right)\right]$$

Thus, $$\frac{1}{S^{\alpha}-2i}\tan 2\frac{s^{\alpha}}{\alpha} = -\frac{1}{2}\left[1 + ie^{2i\frac{s^{\alpha}}{\alpha}}\log\tan\left(\frac{\pi}{4}+\frac{s^{\alpha}}{\alpha}\right)\right] \qquad (2)$$

Replacing i by –i in (2), we get $$\frac{1}{S^{\alpha}+2i}\tan 2\frac{s^{\alpha}}{\alpha} = -\frac{1}{2}\left[1 + ie^{-2i\frac{s^{\alpha}}{\alpha}}\log\tan\left(\frac{\pi}{4}+\frac{s^{\alpha}}{\alpha}\right)\right] \qquad (3)$$

From (1), (2) and (3), we have

$$\text{P.I.} = \frac{1}{4i}\left\{-\frac{i}{2}\left(e^{i2\frac{s^{\alpha}}{\alpha}} + e^{-i2\frac{s^{\alpha}}{\alpha}}\right)\log\tan\left(\frac{\pi}{4}+\frac{s^{\alpha}}{\alpha}\right)\right\} = -\frac{1}{4}\cos 2\frac{s^{\alpha}}{\alpha}\log\tan\left(\frac{\pi}{4}+\frac{s^{\alpha}}{\alpha}\right)$$

$\therefore$ The required solution is y=C.F+P.I.

$$y = l_1\cos 2\frac{s^{\alpha}}{\alpha} + l_2\sin 2\frac{s^{\alpha}}{\alpha} - \frac{1}{4}\cos 2\frac{s^{\alpha}}{\alpha}\log\tan\left(\frac{\pi}{4}+\frac{s^{\alpha}}{\alpha}\right)$$

4.2 Method for finding P.I. when S=$e^{a\frac{s^{\alpha}}{\alpha}}$.

Theorem 4.2.1:

Prove that $\frac{1}{f\left(S^{\alpha}\right)} e^{a\frac{s^{\alpha}}{\alpha}} = \frac{1}{f(a)} e^{a\frac{s^{\alpha}}{\alpha}}$, $f(a) \neq 0$.

Proof:

Let $f\left(S^{\alpha}\right) = S^{\alpha n} + l_1 S^{\alpha(n-1)} + l_2 S^{\alpha(n-2)} + \ldots + l_{n-1}S^{\alpha} + l_n$

But $S^{\alpha} e^{a\frac{s^{\alpha}}{\alpha}} = a e^{a\frac{s^{\alpha}}{\alpha}}$

$S^{2\alpha} e^{a\frac{s^{\alpha}}{\alpha}} = a^2 e^{a\frac{s^{\alpha}}{\alpha}} \ldots S^{\alpha n} e^{a\frac{s^{\alpha}}{\alpha}} = a^n e^{a\frac{s^{\alpha}}{\alpha}}$

$$\therefore f\left(S^{\alpha}\right) e^{a\frac{s^{\alpha}}{\alpha}} = \left(S^{\alpha n} + l_1 S^{\alpha(n-1)} + l_2 S^{\alpha(n-2)} + \ldots + l_{n-1} S^{\alpha} + l_n\right) e^{a\frac{s^{\alpha}}{\alpha}}$$

$$= \left(a^n + l_1 a^{n-1} + l_2 a^{n-1} + \ldots + l_{n-1} a^{n-1} + l_n\right) e^{a\frac{s^{\alpha}}{\alpha}}$$

$$= f\left(a\right) e^{a\frac{s^{\alpha}}{\alpha}}$$

Now, operating both sides by $\frac{1}{f\left(S^{\alpha}\right)}$, we get

$$e^{a\frac{s^{\alpha}}{\alpha}} = \frac{1}{f\left(S^{\alpha}\right)} f\left(a\right) e^{a\frac{s^{\alpha}}{\alpha}}$$

Thus, $\frac{1}{f\left(S^{\alpha}\right)} e^{a\frac{s^{\alpha}}{\alpha}} = \frac{1}{f\left(a\right)} e^{a\frac{s^{\alpha}}{\alpha}}, f\left(a\right) \neq 0$

Example 4.2.2:

Solve $\left(S^{2\alpha} - 3S^{\alpha} + 2\right) y = e^{3\frac{s^{\alpha}}{\alpha}}$ where $S^{\alpha} = \frac{d^{\alpha}}{ds^{\alpha}}$.

Solution:

Here, the auxiliary equation is $S^{2\alpha} - 3S^{\alpha} + 2 = 0$ so that S^{α} =1, 2.

$$\therefore C.F. = \left(l_1 e^{\frac{s^{\alpha}}{\alpha}} + l_2 e^{2\frac{s^{\alpha}}{\alpha}} \right)$$

And P.I. = $\frac{1}{S^{2\alpha} - 3S^{\alpha} + 2} e^{3\frac{s^{\alpha}}{\alpha}} = \frac{1}{(3)^2 - (3.3) + 2} e^{3\frac{s^{\alpha}}{\alpha}} = \frac{1}{2} e^{3\frac{s^{\alpha}}{\alpha}}$

The required solution is $y = l_1 e^{\frac{s^{\alpha}}{\alpha}} + l_2 e^{2\frac{s^{\alpha}}{\alpha}} + \frac{1}{2} e^{3\frac{s^{\alpha}}{\alpha}}$.

Remark 4.2.3:

Prove that when $f\left(a\right) = 0$ i.e. when $f\left(S^{\alpha}\right) = \left(S^{\alpha} - a\right)^n$, then $\frac{1}{\left(S^{\alpha} - a\right)^n} e^{a\frac{s^{\alpha}}{\alpha}} = \frac{\left(\frac{s^{\alpha}}{\alpha}\right)^n}{n!} e^{a\frac{s^{\alpha}}{\alpha}}$.

Proof:

Here, $\frac{1}{f\left(S^{\alpha}\right)} e^{a\frac{s^{\alpha}}{\alpha}} = \frac{1}{\left(S^{\alpha} - a\right)^n} e^{a\frac{s^{\alpha}}{\alpha}}$

$$= \frac{1}{\left(S^{\alpha} - a\right)^{n-1}} \cdot \frac{1}{\left(S^{\alpha} - a\right)} e^{a\frac{s^{\alpha}}{\alpha}}$$

$$=\frac{1}{\left(S^{\alpha}-a\right)^{n-1}}.e^{a\frac{s^{\alpha}}{\alpha}}I_{\alpha}\left(e^{a\frac{s^{\alpha}}{\alpha}}.e^{-a\frac{s^{\alpha}}{\alpha}}\right)$$

$$=\frac{1}{\left(S^{\alpha}-a\right)^{n-1}}.e^{a\frac{s^{\alpha}}{\alpha}}I_{\alpha}\left(1\right)$$

$$=\frac{1}{\left(S^{\alpha}-a\right)^{n-1}}.e^{a\frac{s^{\alpha}}{\alpha}}\int\frac{1}{s^{1-\alpha}}ds$$

$$=\frac{1}{\left(S^{\alpha}-a\right)^{n-1}}.e^{a\frac{s^{\alpha}}{\alpha}}I_{\alpha}\left(1\right)$$

$$=\frac{1}{\left(S^{\alpha}-a\right)^{n-1}}.e^{a\frac{s^{\alpha}}{\alpha}}\left(\frac{s^{\alpha}}{\alpha}\right)$$

$$=\frac{1}{\left(S^{\alpha}-a\right)^{n-2}}.\frac{1}{\left(S^{\alpha}-a\right)}.\left(\frac{s^{\alpha}}{\alpha}\right)e^{a\frac{s^{\alpha}}{\alpha}}$$

$$=\frac{1}{\left(S^{\alpha}-a\right)^{n-2}}.e^{a\frac{s^{\alpha}}{\alpha}}I_{\alpha}\left(\left(\frac{s^{\alpha}}{\alpha}\right)e^{a\frac{s^{\alpha}}{\alpha}}.e^{-a\frac{s^{\alpha}}{\alpha}}\right)$$

$$=\frac{1}{\left(S^{\alpha}-a\right)^{n-2}}.e^{a\frac{s^{\alpha}}{\alpha}}I_{\alpha}\left(\frac{s^{\alpha}}{\alpha}\right)$$

$$=\frac{1}{\left(S^{\alpha}-a\right)^{n-2}}.e^{a\frac{s^{\alpha}}{\alpha}}\int\frac{\frac{s^{\alpha}}{\alpha}}{s^{1-\alpha}}ds$$

$$=\frac{1}{\left(S^{\alpha}-a\right)^{n-2}}.e^{a\frac{s^{\alpha}}{\alpha}}\int\frac{s^{2\alpha-1}}{\alpha}ds$$

$$=\frac{1}{\left(S^{\alpha}-a\right)^{n-2}}.e^{a\frac{s^{\alpha}}{\alpha}}\frac{s^{2\alpha}}{\alpha\left(2\alpha\right)}$$

$$=\frac{1}{\left(S^{\alpha}-a\right)^{n-n}}.e^{a\frac{s^{\alpha}}{\alpha}}\frac{s^{n\alpha}}{\alpha\left(2\alpha\right)\left(3\alpha\right)\ldots\left(n\alpha\right)}$$

$$=\left(\frac{s^{\alpha}}{\alpha}\right)^{n}\frac{1}{n!}e^{a\frac{s^{\alpha}}{\alpha}}$$

Thus, $$\frac{1}{\left(S^{\alpha}-a\right)^{n}}e^{a\frac{s^{\alpha}}{\alpha}}=\frac{\left(\frac{s^{\alpha}}{\alpha}\right)^{n}}{n!}e^{a\frac{s^{\alpha}}{\alpha}}$$

Example 4.2.4:

Solve $\left(S^{2\alpha}-3S^{\alpha}+2\right)y=e^{\frac{s^\alpha}{\alpha}}$ where $S^{\alpha}=\frac{d^{\alpha}}{ds^{\alpha}}$.

Solution:

Here, the auxiliary equation is $S^{2\alpha}-3S^{\alpha}+2=0$ so that S^{α} =1, 2.

$$\therefore C.F.=\left(l_1e^{\frac{s^\alpha}{\alpha}}+l_2e^{2\frac{s^\alpha}{\alpha}}\right)$$

And P.I. = $\frac{1}{S^{2\alpha}-3S^{\alpha}+2}e^{\frac{s^\alpha}{\alpha}}$. Here, $f(1)=0$.

Thus, P.I. = $\frac{1}{\left(S^{\alpha}-1\right)\left(S^{\alpha}-2\right)}e^{\frac{s^\alpha}{\alpha}}=\frac{1}{\left(S^{\alpha}-1\right)\left(1-2\right)}e^{\frac{s^\alpha}{\alpha}}=-\frac{1}{\left(S^{\alpha}-1\right)}e^{\frac{s^\alpha}{\alpha}}=-\frac{s^{\alpha}}{\alpha}e^{\frac{s^\alpha}{\alpha}}$

The required solution is $y=l_1e^{\frac{s^\alpha}{\alpha}}+l_2e^{2\frac{s^\alpha}{\alpha}}-\frac{s^{\alpha}}{\alpha}e^{\frac{s^\alpha}{\alpha}}$.

4.3 Method for Finding P.I. when $S=\sin a\frac{s^{\alpha}}{\alpha}$ or $\cos a\frac{s^{\alpha}}{\alpha}$.

Theorem 4.3.1:

When $f\left(S^{\alpha}\right)$ can be expressed as $\phi\left(S^{2\alpha}\right)$ and $\phi\left(-a^2\right)\neq 0$, then

$$\frac{1}{\phi\left(S^{2\alpha}\right)}\sin a\frac{s^{\alpha}}{\alpha}=\frac{1}{\phi\left(-a^2\right)}\sin a\frac{s^{\alpha}}{\alpha} \text{ and } \frac{1}{\phi\left(S^{2\alpha}\right)}\cos a\frac{s^{\alpha}}{\alpha}=\frac{1}{\phi\left(-a^2\right)}\cos a\frac{s^{\alpha}}{\alpha}.$$

Proof:

By successive differentiation, we have

$$S^{\alpha}\sin a\frac{s^{\alpha}}{\alpha}=a\cos a\frac{s^{\alpha}}{\alpha}$$

$$S^{2\alpha}\sin a\frac{s^{\alpha}}{\alpha}=-a^2\sin a\frac{s^{\alpha}}{\alpha}\Rightarrow\left(S^{2\alpha}\right)^1\sin a\frac{s^{\alpha}}{\alpha}=\left(-a^2\right)^1\sin a\frac{s^{\alpha}}{\alpha} \quad (1)$$

$$S^{3\alpha}\sin a\frac{s^{\alpha}}{\alpha}=-a^3\cos a\frac{s^{\alpha}}{\alpha}$$

$$S^{4\alpha}\sin a\frac{s^{\alpha}}{\alpha}=a^4\sin a\frac{s^{\alpha}}{\alpha}=\left(-a^2\right)^2\sin a\frac{s^{\alpha}}{\alpha} \quad (2).$$

$$S^{2n\alpha}\sin a\frac{s^{\alpha}}{\alpha}=-a^{2n}\sin a\frac{s^{\alpha}}{\alpha}=\left(-a^2\right)^n\sin a\frac{s^{\alpha}}{\alpha} \quad (3)$$

Let $\phi\left(S^{2\alpha}\right)=\left(S^{2\alpha}\right)^n+a_1\left(S^{2\alpha}\right)^{n-1}+\ldots+a_{n-1}\left(S^{2\alpha}\right)^1+a_n$ (4)

From (1), (2),…, (3) and (4), it follows that

$$\phi\left(S^{2\alpha}\right)\sin a\frac{s^{\alpha}}{\alpha}=\phi\left(-a^2\right)\sin a\frac{s^{\alpha}}{\alpha} \quad (5)$$

Operating both sides of (5) by $\frac{1}{\phi\left(S^{2\alpha}\right)}$, we have

$$\sin a\frac{t^{\alpha}}{\alpha}=\frac{1}{\phi\left(T^{2\alpha}\right)}\phi\left(-a^{2}\right)\sin a\frac{t^{\alpha}}{\alpha}$$

Dividing both sides by $\phi\left(-a^{2}\right)\neq 0$, we get

$$\frac{1}{\phi\left(S^{2\alpha}\right)}\sin a\frac{s^{\alpha}}{\alpha}=\frac{1}{\phi\left(-a^{2}\right)}\sin a\frac{s^{\alpha}}{\alpha}\text{, where }\phi\left(-a^{2}\right)\neq 0.$$

Similarly, we can obtain the result for cosine function: $\frac{1}{\phi\left(S^{2\alpha}\right)}\cos a\frac{s^{\alpha}}{\alpha}=\frac{1}{\phi\left(-a^{2}\right)}\cos a\frac{s^{\alpha}}{\alpha}$.

Example 4.3.2:

Solve $\left(S^{2\alpha}+1\right)y=\cos 2\frac{s^{\alpha}}{\alpha}$ where $S^{\alpha}=\frac{d^{\alpha}}{ds^{\alpha}}$.

Solution:

Here, the auxiliary equation is given by $\left(S^{2\alpha}+1\right)=0$ so that $S^{\alpha}=\pm i$.

$$\therefore C.F.=l_{1}\cos\frac{s^{\alpha}}{\alpha}+l_{2}\sin\frac{s^{\alpha}}{\alpha}\ .$$

$$\text{P.I.}=\frac{1}{\left(S^{2\alpha}+1\right)}\cos 2\frac{s^{\alpha}}{\alpha}=\frac{1}{\left((-2)^{2}+1\right)}\cos 2\frac{s^{\alpha}}{\alpha}=-\frac{1}{3}\cos 2\frac{s^{\alpha}}{\alpha}$$

Thus, the required solution is $y=l_{1}\cos\frac{s^{\alpha}}{\alpha}+l_{2}\sin\frac{s^{\alpha}}{\alpha}-\frac{1}{3}\cos 2\frac{s^{\alpha}}{\alpha}$.

4.4 When $S=e^{a\frac{s^{\alpha}}{\alpha}}V$, where V is a function of s.

Theorem 4.4.1:

Prove that $\frac{1}{f\left(S^{\alpha}\right)}e^{a\frac{s^{\alpha}}{\alpha}}V=S=e^{a\frac{s^{\alpha}}{\alpha}}\frac{1}{f\left(S^{\alpha}+a\right)}V$, V being a function of t.

Proof:

By successive differentiation, we have

$$S^{\alpha}\left(e^{a\frac{s^{\alpha}}{\alpha}}V\right)=e^{a\frac{s^{\alpha}}{\alpha}}S^{\alpha}V+ae^{a\frac{s^{\alpha}}{\alpha}}V=e^{a\frac{s^{\alpha}}{\alpha}}\left(S^{\alpha}+a\right)V$$

$$S^{2\alpha}\left(e^{a\frac{s^{\alpha}}{\alpha}}V\right)=e^{a\frac{s^{\alpha}}{\alpha}}S^{2\alpha}V+ae^{a\frac{s^{\alpha}}{\alpha}}S^{\alpha}V+ae^{a\frac{s^{\alpha}}{\alpha}}S^{\alpha}V+a^{2}e^{a\frac{s^{\alpha}}{\alpha}}V$$

$$=e^{a\frac{s^{\alpha}}{\alpha}}\left(S^{2\alpha}+2aS^{\alpha}+a^{2}\right)V$$

$$=e^{a\frac{s^{\alpha}}{\alpha}}\left(S^{\alpha}+a\right)^{2}V$$

Similarly, $S^{2\alpha}\left(e^{a\frac{s^{\alpha}}{\alpha}}V\right)=e^{a\frac{s^{\alpha}}{\alpha}}\left(S^{\alpha}+a\right)^{3}V$

$$S^{\alpha n}\left(e^{a\frac{s^\alpha}{\alpha}}V\right)=e^{a\frac{s^\alpha}{\alpha}}\left(S^\alpha+a\right)^n V$$

$$f\left(S^\alpha\right)e^{a\frac{s^\alpha}{\alpha}}V=e^{a\frac{s^\alpha}{\alpha}}f\left(S^\alpha+a\right)V \qquad (1)$$

Result (1) is true for any function of s, taking $\frac{1}{f\left(S^\alpha+a\right)}V$ in place of V in (1), we have

$$f\left(S^\alpha\right)\left\{e^{a\frac{s^\alpha}{\alpha}}\frac{1}{f\left(S^\alpha+a\right)}V\right\}=e^{a\frac{s^\alpha}{\alpha}}f\left(S^\alpha+a\right)\left\{\frac{1}{f\left(S^\alpha+a\right)}V\right\}$$

$$\Rightarrow e^{a\frac{s^\alpha}{\alpha}}V=f\left(S^\alpha\right)\left\{e^{a\frac{s^\alpha}{\alpha}}\frac{1}{f\left(S^\alpha+a\right)}V\right\}$$

$$\Rightarrow \frac{1}{f\left(S^\alpha\right)}e^{a\frac{s^\alpha}{\alpha}}V=e^{a\frac{s^\alpha}{\alpha}}\frac{1}{f\left(S^\alpha+a\right)}V$$

Example 4.4.2:

Solve $\left(S^{2\alpha}+3S^\alpha+2\right)y=e^{2\frac{s^\alpha}{\alpha}}\sin\frac{s^\alpha}{\alpha}$, where $S^\alpha=\frac{d^\alpha}{ds^\alpha}$.

Solution:

Here the auxiliary equation is given by $\left(S^{2\alpha}+3S^\alpha+2\right)=0\Rightarrow S^\alpha=-1,-2$.

$$\therefore C.F.=l_1e^{-\frac{s^\alpha}{\alpha}}+l_2e^{-2\frac{s^\alpha}{\alpha}}$$

P.I.=

$$\frac{1}{\left(S^{2\alpha}+3S^\alpha+2\right)}e^{2\frac{s^\alpha}{\alpha}}\sin\frac{s^\alpha}{\alpha}=e^{2\frac{s^\alpha}{\alpha}}\frac{1}{\left(\left(S^\alpha+2\right)^2+3\left(S^\alpha+2\right)+2\right)}\sin\frac{s^\alpha}{\alpha}=e^{2\frac{s^\alpha}{\alpha}}\frac{1}{\left(S^{2\alpha}+7S^\alpha+12\right)}\sin\frac{s^\alpha}{\alpha}$$

$$=e^{2\frac{s^\alpha}{\alpha}}\frac{1}{\left(-1^2+7S^\alpha+12\right)}\sin\frac{s^\alpha}{\alpha}=e^{2\frac{s^\alpha}{\alpha}}\frac{1}{\left(11+7S^\alpha\right)}\sin\frac{s^\alpha}{\alpha}=e^{2\frac{s^\alpha}{\alpha}}\left(11-7S^\alpha\right)\frac{1}{\left(11+7S^\alpha\right)\left(11-7S^\alpha\right)}\sin\frac{s^\alpha}{\alpha}$$

$$=e^{2\frac{s^\alpha}{\alpha}}\left(11-7S^\alpha\right)\frac{1}{\left(121-49S^{2\alpha}\right)}\sin\frac{s^\alpha}{\alpha}=e^{2\frac{s^\alpha}{\alpha}}\left(11-7S^\alpha\right)\frac{1}{\left(121-49\left(-1^2\right)\right)}\sin\frac{s^\alpha}{\alpha}=\frac{e^{2\frac{s^\alpha}{\alpha}}}{170}\left(11-7S^\alpha\right)\sin\frac{s^\alpha}{\alpha}$$

$$\frac{1}{170}e^{2\frac{s^\alpha}{\alpha}}\left(11\sin\frac{s^\alpha}{\alpha}-7S^\alpha\sin\frac{s^\alpha}{\alpha}\right)=\frac{1}{170}e^{2\frac{s^\alpha}{\alpha}}\left(11\sin\frac{s^\alpha}{\alpha}-7\cos\frac{s^\alpha}{\alpha}\right)$$

Thus, the required solution is $y=l_1e^{-\frac{s^\alpha}{\alpha}}+l_2e^{-2\frac{s^\alpha}{\alpha}}+\frac{1}{170}e^{2\frac{s^\alpha}{\alpha}}\left(11\sin\frac{s^\alpha}{\alpha}-7\cos\frac{s^\alpha}{\alpha}\right)$

5. Applications:

Section 5 utilizes our work to solve real-world problems and points out how they are easy to comprehend, compute, and thus applied.

5.1 Mass-Spring-Damper System [13]:

The conformable fractional differential equation for the motion of the mass on a spring (mass-spring-damper system) is given by:

$$\frac{m}{\sigma^{2(1-\alpha)}}\frac{d^{2\alpha}x(t)}{dt^{2\alpha}}+\frac{\beta}{\sigma^{1-\alpha}}\frac{d^{\alpha}x(t)}{dt^{\alpha}}+kx(t)=0 \quad ,0<\alpha\le 1 \tag{5.1}$$

$$\frac{m}{\sigma^{2(1-\alpha)}}\frac{d^{2\alpha}x(t)}{dt^{2\alpha}}+\frac{\beta}{\sigma^{1-\alpha}}\frac{d^{\alpha}x(t)}{dt^{\alpha}}+kx(t)=F(t) \quad ,0<\alpha\le 1 \tag{5.2}$$

Particular cases:

When $\beta=0$, the above equations reduces to

$$\frac{m}{\sigma^{2(1-\alpha)}}\frac{d^{2\alpha}x(t)}{dt^{2\alpha}}+kx(t)=0 \quad ,0<\alpha\le 1 \tag{5.3}$$

$$\frac{m}{\sigma^{2(1-\alpha)}}\frac{d^{2\alpha}x(t)}{dt^{2\alpha}}+kx(t)=F(t) \quad ,0<\alpha\le 1 \tag{5.4}$$

When $m=0$, the above equations reduces to

$$\frac{\beta}{\sigma^{1-\alpha}}\frac{d^{\alpha}x(t)}{dt^{\alpha}}+kx(t)=0 \quad ,0<\alpha\le 1 \tag{5.5}$$

$$\frac{\beta}{\sigma^{1-\alpha}}\frac{d^{\alpha}x(t)}{dt^{\alpha}}+kx(t)=F(t) \quad ,0<\alpha\le 1 \tag{5.6}$$

Where $\frac{d^{\alpha}}{dt^{\alpha}}$ is the conformable fractional derivative, $m=$ mass, $\beta=$ damping constant, $k=$ spring constant and $F(t)=$ external impressed force, $\sigma=$ cosmic time

Remark 5.1.1:

a) If the damping constant, $\beta=0$, then the system is called undamped otherwise damped.

b) If the external impressed force, $F(t)=0$, the system is called free otherwise forced.

The solution for Equation (5.3) with $0<\alpha\le 1$, $\alpha=\frac{\sigma}{\sqrt{\frac{m}{k}}}$, $0<\sigma\le\sqrt{\frac{m}{k}}$, is given by

$$x(t)=l_1\cos\mu\frac{t^{\alpha}}{\alpha}+l_2\sin\mu\frac{t^{\alpha}}{\alpha} \text{ , where } \mu^2=\frac{k\sigma^{2(1-\alpha)}}{m}$$

Suppose that the mass was initially displaced a distance x_0 from its equilibrium position and released from that point with initial velocity v_0. Then, using the initial conditions $x(0)=x_0, x'(0)=v_0$, the above eqn. can be written as

$$x(t)=x_0\cos\mu\frac{t^{\alpha}}{\alpha}+\frac{v_0}{\mu}\sin\mu\frac{t^{\alpha}}{\alpha}$$

Again, for a better understanding, we can write $x(t)=C\cos\left(\mu\frac{t^{\alpha}}{\alpha}+\phi\right)$

Where $C=\sqrt{\left(\frac{v_0}{\mu}\right)^2+(x_0)^2}$, $\frac{\frac{v_0}{\mu}}{C}=-\sin\phi$ and $\frac{x_0}{C}=\cos\phi$.

Remark 5.1.2:

For $\alpha = 1$, $\mu = \sqrt{\frac{k}{m}}$ we get, $x(t) = C\cos\left(\sqrt{\frac{k}{m}}t + \phi\right)$, which is the well known solution of the corresponding integer differential equation.

Equation (5.4) represents forced, undamped motion, solution can be found by using the corresponding homogeneous counterpart and its particular integral. In case of a periodic external impressed force, $F(t) = F_0 \cos\omega\frac{t^\alpha}{\alpha}$, for all $t \geq 0$, the particular integral is $\frac{\sigma^{2(1-\alpha)}}{m(\mu^2 - \omega^2)} F_0 \cos\omega\frac{t^\alpha}{\alpha}$. Thus, the complete solution is

$$x(t) = C\cos\left(\mu\frac{t^\alpha}{\alpha} + \phi\right) + \frac{\sigma^{2(1-\alpha)}}{m(\mu^2 - \omega^2)} F_0 \cos\omega\frac{t^\alpha}{\alpha}$$

The solution of eqn. (5.5) with $\alpha = \frac{\sigma k}{m}$ and $0 < \sigma \leq \frac{k}{m}$ is given by

$x(t) = x_0 e^{-\mu\frac{t^\alpha}{\alpha}}$ where $\mu = \frac{k\sigma^{1-\alpha}}{\beta}$ and for initial condition $x(0) = x_0$.

Remark 5.1.3:

For $\alpha = 1$, reduces to $x(t) = x_0 e^{\frac{-kt}{\beta}}$, which is the well known solution of the corresponding integer differential equation.

Now, the solution of the forced equation (5.6) is given as the sum of solution of the homogeneous equation and particular integral. In case, $F(t) = F_0 \cos\omega\frac{t^\alpha}{\alpha}$.

$$P.I. = \frac{1}{(S^\alpha + \mu)} \frac{\sigma^{1-\alpha}}{\beta} F_0 \cos\omega\frac{t^\alpha}{\alpha} = \frac{\sigma^{1-\alpha}}{\beta} F_0 \frac{(S^\alpha - \mu)}{(S^{2\alpha} - \mu^2)} \cos\omega\frac{t^\alpha}{\alpha}$$

$$= \frac{\sigma^{1-\alpha}}{\beta} F_0 \frac{1}{(\omega^2 + \mu^2)} \left\{\omega\sin\omega\frac{t^\alpha}{\alpha} + \mu\cos\omega\frac{t^\alpha}{\alpha}\right\}$$

$x(t) = x_0 e^{-\mu\frac{t^\alpha}{\alpha}} + \frac{\sigma^{1-\alpha}}{\beta} F_0 \frac{1}{(\omega^2 + \mu^2)} \left\{\omega\sin\omega\frac{t^\alpha}{\alpha} + \mu\cos\omega\frac{t^\alpha}{\alpha}\right\}$, where $\mu = \frac{k\sigma^{1-\alpha}}{\beta}$.

Solution for equation (5.1) with $\beta > 0, k > 0$ can be obtained as follows

$$\frac{d^{2\alpha}x(t)}{dt^{2\alpha}} + \frac{\beta\sigma^{1-\alpha}}{m}\frac{d^\alpha x(t)}{dt^\alpha} + \frac{k\sigma^{2(1-\alpha)}}{m}x(t) = 0$$

$(S^{2\alpha} + 2bS^\alpha + \mu^2)x(t) = 0$, where $2b = \frac{\beta\sigma^{1-\alpha}}{m}, \mu^2 = \frac{k\sigma^{2(1-\alpha)}}{m}$

$$S^\alpha = -b \pm \sqrt{b^2 - \mu^2}$$

The displacements of mass become negligible over a long period of time, since each solution contains the term $e^{-b\frac{t^\alpha}{\alpha}}, b > 0$. Three different cases arise depending upon the nature of these roots, which in turn depends upon the sign of $b^2 - \mu^2$.

Case I: Motion of an Over Damped System

Here we consider the case in which $b^2 - \mu^2 > 0$, β □ k. The corresponding solution is

$$x(t) = e^{-b\frac{t^\alpha}{\alpha}} \left\{ l_1 e^{\frac{t^\alpha}{\alpha}\sqrt{b^2-\mu^2}} + l_2 e^{-\frac{t^\alpha}{\alpha}\sqrt{b^2-\mu^2}} \right\}.$$

Case II: Motion of a Critically Damped System

For $b^2 - \mu^2 = 0$, the system is said to be critically damped because a slight decrease in the damping force would result in oscillatory motion and the solution is given by $x(t) = e^{-b\frac{t^\alpha}{\alpha}} \left(l_1 + l_2 \frac{t^\alpha}{\alpha} \right)$.

Case III: Motion of an Under-Damped System

For $b^2 - \mu^2 < 0$, β □ k. The corresponding solution is

$$x(t) = e^{-b\frac{t^\alpha}{\alpha}} \left\{ l_1 \cos \frac{t^\alpha}{\alpha} \sqrt{\mu^2 - b^2} + l_2 \sin \frac{t^\alpha}{\alpha} \sqrt{\mu^2 - b^2} \right\}.$$

Similarly, we can find a solution for equation (5.2) by using the solution of the corresponding homogeneous equation and finding the particular solution by using the methods as mentioned in this text.

5.2 Gompertz Model [11]:

The original formulation of Gompertz's model arises from the following evolution equation [11],

$\frac{1}{V}\frac{dV}{dt} = \eta - \beta \ln\left(\frac{V}{V_0}\right)$, where η and β are kinetic parameters which are always real numbers, $V = V(t)$ is the volume of a tumor at time t and $V_0 = V(0)$ is the initial volume of the tumor at a time $t = 0$.

Using the transformation, $y = \ln\left(\frac{V}{V_0}\right)$ the above equation becomes

$$\frac{dy}{dt} = \eta - \beta y$$

With initial condition $y(0) = 0$

Thus, the fractional counterpart of the above equation can be written as

$$\frac{d^\alpha y(t)}{dt^\alpha} = \eta - \beta y(t)$$

$$y(0) = 0$$

$\left(S^\alpha + \beta\right) y(t) = \eta$, where β is the real constant.

Using the methods mentioned in section 3 and 4, we obtain the solution of the Gompertz equation:

$$\Rightarrow V(t) = V_0 \exp\left\{ \frac{\eta}{\beta}\left(1 - e^{-\beta\frac{t^\alpha}{\alpha}}\right) \right\}$$

Remark 5.2.1:

For $\alpha = 1 \Rightarrow V(t) = V_0 \exp\left\{ \frac{\eta}{\beta}\left(1 - e^{-\beta t}\right) \right\}$, which is the well-known Gompertz Law.

6. Conclusion:

Our work generalizes the concept of linear differential equations to fractional-order operators and addresses it analytically, within the framework of the limit-based definition of conformable fractional derivatives. We produce the relationship between the conformable fractional integral and usual integrals. This simplifies the process of fractional integration. By knowing the usual integrals we can easily find the corresponding fractional ones. We introduced the concept of LFDEs with constant coefficients and further obtained the method to find the general solution and particular integral of these equations for cases with different types of roots and functions. Further, we implemented our findings to some real-world problems like the mass-spring-damper system and the well-known Gompertz Law for population and tumor growth. The results shown by our method are in complete harmony with the well-known results of these systems. Moreover, the computation is much easier as the general solutions and particular integrals derived are on similar lines as that of the ordinary derivatives. Hence, we believe that future work on conformable fractional differential equations will be stimulated by our study.

7. References:

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