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GRACEFUL AND GRACEFUL LABELING OF GRAPHS

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Abstract:

The concept of a graph labeling is an active area of research in graph theory which has rigorous applications in coding theory, communication networks, optimal circuits layouts and graph decomposition problems.Graph labeling were first introduced in the late 1960s and have been motivated by practical problems. In the intervening years variety of graph labeling techniques have been studied and the subject is growing exponentially.

Key Words: Graph, Decomposition & Labeling.

Introduction:

A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions. Graph labeling is one of the most interesting problems in graph theory and the serve as useful models for a broad range of applications such as coding theory, X-ray crystallography, circuit designs ect. Graph labeling was first introduced in the late 1960s and have been motivated by practical problems. In the intervening years variety of graph labeling techniques have been studied and the subject is growing exponentially. For more details one may refer the survey article (1) by J. A. Gallian. The graceful labeling methods were introduced by Rosa (6) in 1967. A graph G with p and q edges is graceful if $f$is an injection from the vertices of G to the set $\{0,1,\ldots.p\}$ such that, when each edge uv is assigned the label $|f(u) - f(v)|$, the resulting edge labels are distinct.

Basic Definition:

Definition 1: A graph is a tuple G = (V, E) where V is a (finite) set of vertices and E is a finite collection of edges. The set E contains elements from the union of the one and two elements subset of V. That is, each edge is either a one or two element subset of V.

Definition 2: Let G = (V, E). A graph H = (V^1, E^2) is a subgraph of G if $V^1 \subseteq V$ and $E^1 \subseteq E$. The subgraph H is proper if $V^1 \subseteq V$ or $E^1 \subseteq E$.

Definition 3: A graph G = (V, E) is a simple graph. If G has no edges that are self – loops and if E is a subset of two element subset of V;

(i.e) G is not a multi – graph.

Definition 4: An edge of a graph that joins a vertex to itself is called a loop. A loop is an edge e = V_iV_i.

Definition 5: If two vertices of a graph are joined by more than one edge that these edges are called multiple edges.

Definition 6: If G = (V, E) is a graph and v ∈ V and e = {v}, then edge e is called a self – loop. That is, any edge that is a single element subset of V is called a self – loop.

Definition 7: A graph G = (V, E) is a multigraph if there are two edgese_1 and e_2 in E so that e_1 and e_2 are equal assets. That is, there are two vertices v_1 and v_2 in V so that$e_1 = e_2 = \{v_1,v_2\}$.

Definition 8: A directed graph (diagraph) is a tuple G = (V, E) where V is a (finite) set of vertices and E is a collection of elements contained in V×V. That is, E is a collection of ordered pairs of vertices. The edges in E are called directed edges to distinguish them from those edges in graph.

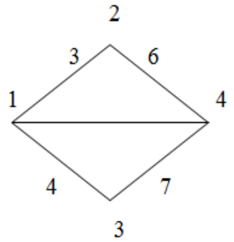
Definition 9: A weighted graph is a graph in which each edge is assigned aweight, which is a positive number.

Definition 10: A graph G is said to be strong vertex graceful if there exists a bijective mapping f : V(G) → {1, 2,…,n} such that for the induced labelingf^+: E(G) → $\mathbb{N}$ defined by $f^+\big((u,v)\big) = f(u) + f(v), f^+(E(G))$ consists of consecutive integers.

On Strong Vertex Graceful Graphs:

Definition 11: A graph G is said to be strong vertex graceful if there exists a bijective mapping f: V(G)→{1,2,…,n} such that for the labeling f^+: E(G) → N defined by $f^+((u, v)) = f(u) + f(v)$, f + (E(G)) consists of consecutive integer

Note: If we replace N by Z_n, where n is the number of edges in G, we get a bijective map. Lee, Pan and Tsai (4) called such graphs vertex graceful. They observe that the class of vertex graceful graphs properly contains the super edge-magic graphs and strong vertex graceful graphs are super edge - magic. They provided vertex graceful and strong vertex graceful labeling for various graphs of small orders. For example the following graph G of order 4 and size 5 is strong vertex graceful



Note: In a strong vertex graceful graph with n vertices $v_{1,}v_{2,....,}v_n$ labeled 1, 2,…., n, the maximum value ofan edge will be 2_n-1 (if there is an edge $v_{n-1}v_n$) and the maximum value of an edge will be 3 (if there is an edge v_1v_2). Hence the maximum number of edges in a strong vertex graceful graphs with *n* vertices will be 2_n-3. The concept of energy of a graph was introduced by I. Gutman [2] in 1978. Let G be a graph with *n* vertices and *m* edges and let A=(a_{ij}) be the adjacency matrix of the graph. The eigen values $\lambda_1, \lambda_2, \dots \lambda_n$ of A, assumed in non increasing order, are the eigen values of the graph G. The set of eigen values of G, space G, is independent of labeling of the vertices of G. As A is real symmetric, the eigen values of G are real with sum equal to zero. Thus $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n, \lambda_1 + \lambda_2 + \cdots + \lambda_n = 0$. The energy E(G) of G is defined to be the sum of the absolute values of the eigen values of G. That is,

$$E(G) = \sum_{i=0}^{n} |\lambda_i|.$$

The laplacian energy of a graph G has recently been defined by Gutman and B. Zhou [3]. Let G be a graph with *n* vertices and *m* edges. The Laplacian matrix of the graph G, denoted by L = (L_{ij}) , is a square matrix of order *n* whose elements are defined as

$$L_{ij} = \begin{cases} \delta_i, \text{if } i = j \\ -1, \text{if } i \neq j \\ 0, \text{if } i \neq j \end{cases}$$

whereδ_i = degree of vertex v_i. Let $\mu_1, \mu_2, \dots \mu_n$ be the laplacian eigen values of G. laplacian energy LE(G) of G is defined as

$$LE(G) = \sum_{i=0}^{n} \left|\mu_i - \frac{2m}{n}\right|.$$

The laplacian - energy - like invariant of a graph and showed that there is a great deal of analogy between the properties of E(G) and the Laplacian - energy - like invariant graph LEL(G). In fact they have first introduced the auxiliary eigen values $\rho_i, i = 1,2, \dots, n$are Laplacian eigen values of the graph G. Let G(n, m)be a graph with *n* vertices and *m* edges and let the Laplacian eigen values of G are $\mu_{1,}\mu_{2,\dots,}\mu_n$.

Then the laplacian-energy-like invariant of G is defined by LET (G) = $\sum_{i=0}^{n} \rho_i = \sum_{i=0}^{n} \sqrt{\mu_i}$. In this paper we find the energy, the Laplacian energy and the Laplacian-energy-like invariant of an $(n, 2n-3)$ strong vertex graceful graph. Also we find the cardinality of the set of all $(n, 2n-3)$ strong vertex graceful graphs.

Theorem 1:

There exists a (n, 2n - 3) graph which is strong vertex graceful.

Proof:

Let the vertices of the graph be $v_1, v_2, \dots, v_n$. we label the vertices as follows. v_i = i, $1 \leq i \leq n$.

Next we connect v_1 to v_k, $2 \leq k \leq n$ and $v_n$to v_k, $2 \leq k \leq n-1$. Then the value of the edge $v_1v_k$is k + 1 for $2 \leq k \leq n$ and that of the edge v_nv_k isn + kfor $2 \leq k \leq n-1$. Thus the 2n - 3 edges take the consecutive values 3, 4,…, 2n - 1 and the (n,2n - 3) graph becomes strong vertex graceful.

Theorem 2:

Let G be a (n, 2n - 3) strong vertex graceful graph with vertices $v_1, v_2, \dots v_n$labeled as $v_i = i, 1 \leq i \leq n$and the edges are obtained by joining v_1 to v_k, $2 \leq k \leq n$ andv_n to v_k, $2 \leq k \leq n-1$.Then the energy of G is given by E(G)=1+$\sqrt{8_n - 15}$.

Proof:

The adjacency matrix of graph

$$\begin{pmatrix} 0 & 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & 0 & \dots & 0 & 1 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 1 & 0 & 0 & \dots & 0 & 1 \\ 1 & 1 & 1 & \dots & 1 & 0 \end{pmatrix}$$

The characteristic polynomial of the matrix is given by $\phi(G:\lambda) = \lambda^n - (2n-3)\lambda^{n-2} - (2n-4)\lambda^{n-3}$

$$= \lambda^{n-3}[\lambda^3 - (2n-4)].$$

Hence the eigenvalues of the adjacency matrix are $\lambda_1 = \frac{-1+\sqrt{8n-15}}{2}$

$$\lambda_2 = -1$$

$$\lambda_3 = \frac{-1-\sqrt{8n-15}}{2}$$

$$\lambda_k = 0, 4 \leq k \leq n.$$

Therefore, the energy is given by, $E_{SVG}(G) = \sum_{i=0}^{n}|\lambda_i| = 1 + \sqrt{8n-15}$.

Theorem 3: Let G be a (n, 2n - 3) strong vertex graceful graph as in previous theorem. Then the laplacian energy of G is given by LE(G) = $\frac{4(n^2-4n+6)}{n}$

Proof:

The laplacian matrix of G is,

$$\begin{pmatrix} n-1 & -1 & -1 & \dots & -1 & -1 \\ -1 & 2 & 2 & \dots & 0 & -1 \\ -1 & 0 & 2 & \dots & 0 & -1 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ -1 & 0 & 0 & \dots & 2 & -1 \\ -1 & -1 & -1 & \dots & -1 & n-1 \end{pmatrix}$$

The characteristic polynomial of the graph is (G; μ) = $\mu^n - (4n-6)\mu^{n-1} + \dots + (-1)^n \mu n^2 2^{n-3}$

$$= \mu(\mu^{n-1} - (4n-6)\mu^{n-2} + \dots + (-1)^n n^2 2^{n-3})$$

Hence the eigen values of the laplacian matrix are $\mu_i = n \text{ for } 1 \leq i \leq 2$

$$\mu_i = 2 \text{ for } 3 \leq i \leq n-1 \text{ and } \mu_n = 0$$

Therefore, the laplacian energy is given by L(E) = $\sum_{i=1}^{n}\left|\mu_i - \frac{2m}{n}\right| = \frac{4(n^2-4n+6)}{n}$.

Edge Graceful Labeling of Some Trees:

Definition 12: All graphs in this paper are finite, simple and undirected. The symbols V(G)and E(G) will denote the vertex set and edge set of a graph G. The cardinality of the edge is called the size of G. A graph with p vertices and q edges is called a (p, q) graph.

Note: A labeling of a graph G is an assignment foflabels to thevertices ofG that induces for each edge x y, a label depending on the vertex labels f(x) and f(y). Let G be a graph with q edges. Let f be an injection from the vertices of G to set {0, 1, 2,…, q} is called a graceful labeling of G if when we assign to each edge x y the label|f(x) − f(y)|. The resulting edge labels are distinct. A graph G with q edges and p vertices is said to be edge graceful if there exists a bijection f from the edge set to the set {1,2,…,q} so that the induced mapping the vertex set to the set{0,1,2,…,p - 1} given by $f^+(x) = \sum\{f(xy/xy \in E(G)\}$ (mod p) is a bijection. The necessary condition for a graph to beedgegraceful is q (q + 1) ≡ 0 or p / 2 (mod p). With this condition one can verify that even cycles, andpaths of even length are not edge graceful. But whether trees of odd order are edge graceful is still open.On attempting to move towards this conjecture, in this paper we checked it for a special type of trees called bi star graph. Here again the odd order trees turn to be edge graceful graph. Thus it confirms that the conjecture is moving towards affirmative.

Definition 13: The graph $B_{n,m}$ is defined as the graph obtained by joining the center u of the star $k_{1,n}$ and the center of another star $k_{1,m}$ to a new vertex w.The number of vertices in the graph $B_{n,m}$ is p = n + m + 3 and the number of edges q = n + m + 2.

Theorem 4:

The bistar graph $B_{n,m}$ for n ≠ m where n, m even is an edge graceful graph.

Proof:

Let $\{w, v, v^1, v_1, v_2 \dots v_n, v_{n+1} \dots v_{n+m}\}$ be the vertices of bistar and edges e_i are defined as follows

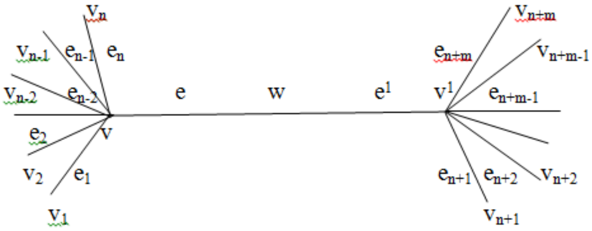


Figure 1: $B_{n,m}$ with ordinary labeling

e = (w, v); $e^1 = (w, v^1)$; $e_i = (v, v_i)$; for i = 1,2,3…..n and $e_i = (v^1, v)$ for i= n + 1, n + 2,…, n + m.

Consider the Diophantine equation $x_1+x_2 = p$. The solutions are of the form (t, p - t) where $1 \leq t \leq q/2$. There will be q/2 pair of solutions. With these pairs label the edges of $K_{1,n}$ and $K_{1,m}$ by the co-ortinates of the pair in any order so that adjacent edges receives the co-ordinates of the pairs. Also, label

$$f^{+}(w) = 0;$$
$$f^{+}(v^{1}) = 1;$$
$$f^{+}(v) = q.$$

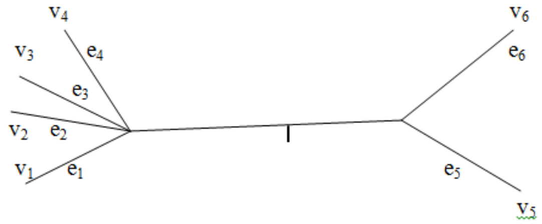
Now the pendent vertices will have labels of the edges with which they are incident and they are distinct. Hence the graph $B_{n,m}$ & $n \neq m$, even is edge graceful.

Illustration:

Consider the bistargraph $B_{4,2}$. Here p = 9

Consider the Diophantine equation $x_1 + x_2 = p = 9$.

The pair of solution are: (t, p - t)



Where t ∈ (1, q/2): (1, 8),(2, 7), (3, 6), (4, 5). We label the edges e &e^1 as follows: f(e) = q =8,f(e^1) =1. Pair of solutions labels the edges of $K_{4,2}$ & $K_{1,2}$by the co-ordinate of the pairs. The edge graceful labeling of bistar $B_{4,2}$.

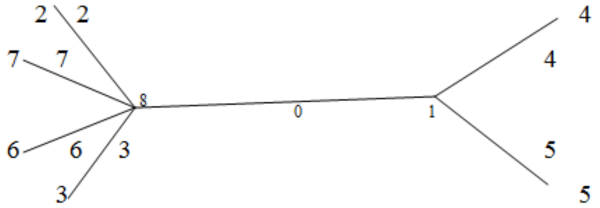


Figure: Edge graceful labeling of $B_{4,2}$

Graceful and Odd Graceful Labeling of Some Trees:

We begin with simple, finite, connected and undirected graph G = (V(G), E(G))

$$|V(G)| = p$$
$$|E(G)| = q$$

Definition 14: If the vertices are assigned values subject to certain conditions then it is known as graph labeling.

Definition 15: A function fis called graceful labeling of a graph G if $f : V(G) \rightarrow \{0, 1, 2, \dots, q\}$is injective and the induced function, $f^*: E(G) \rightarrow \{1, 2, \dots, q\}$defined as $f^*(e = uv) = |f(u) - f(v)|$ is bijective. The graph which admits graceful labeling is called a graceful graph.

Definition 16: A function f is called odd graceful labeling of a graph G if $f : V(G) \rightarrow \{0, 1, 2, \dots, 2q - 1\}$is injective and the induced function, $f^* : E(G) \rightarrow \{1,3, \dots, 2q - 1\}$defined as$f^*(e = uv) = |f(u) - f(v)|$ is bijective. The graph which demits odd graceful labeling is called an odd graceful graph.

Definition 17: For a graph G the splitting graph S' of G is obtained by adding a new vertex v' corresponding to each vertex *v* of G such that N(v) = N(v') .

Definition 18: The shadow graph $D_2(G)$ of a connected graph *G* is constructed by taking two copies of G' and G''.Join each vertex u' in G'' to the neighbours of the corresponding vertex v' in G''.

Definition 19: For a simple connected graph G the square of graph G is denoted by $G^2$and defined as the graph with the same vertex set as ofG and two vertices are adjacent in G^2 if they are at a distance 1 or 2 apart in G.

Theorem 5:

$B_{n,n}^2$ is a graceful graph.

Proof:

Consider $B_{n,n}$ with the vertex set $\{u, v, u_i, v_i, 1 \leq i \leq n\}$ where u_i, v_i are the pendant vertices.Let G be the graph $B_{n,n}^2$ then $|V(G)| = 2n + 2$ and $|E(G)| = 4n + 1$. We define the vertex labeling $f: V(G) \rightarrow \{0,1,2, \dots, 4n + 1\}$ as follows. f(v) =0, f(u) = 4n + 1, $f(v_i) = i$;$1 \leq i \leq n$, $f(u_i) = f(v_n) + i$; $1 \leq i \leq n$

The vertex function f defined above induces a bijective edge function $f^* : E(G) \rightarrow \{1, 2, 3, \dots, 4n + 1\}$. Thus fis graceful labeling ofG =$B_{n,n}^2$. Hence,$B_{n,n}^2$is a graceful graph.

Theorem 6:

$S'(B_{n,n})$ is a graceful graph.

Proof:

Consider $B_{n,n}$ with the vertex set $\{u, v, u_i, v_i, 1 \leq i \leq n\}$ where u_i, v_i are the pendent vertices.In order to obtain $S'(B_{n,n})$, and u', v', u_i', v_i' vertices corresponding to u, v, u_i, v_i where, $1\leq i \leq n$.If $G = S'(B_{n,m})$ then $|V(G)| = 4(n+1)$ and $|E(G)| = 3(2n+1)$. To define the vertex labeling,

$$f: V(G) \rightarrow \{0,1,2,3, \dots, 6n+3\}.$$

We consider the following two cases.

Case 1: n is even.

$$f(u) = 2n-1, f(v) = 0, \ f(u') = 2n, \ f(v') = 1, fu_1' = 4n,$$
$$f(u_{1+i}') = n-1+i; 1\leq i \leq n-1$$
$$f(u_i) = 1+2i; 1\leq i \leq \frac{n-2}{2}$$
$$f\left(u_{\frac{n}{2}-1+i}\right) = 5n+4-2i; \ 1 \leq i \leq \frac{n+2}{2} \ f(v'_{1+i}) = 6n+3-i; \ 1 \leq i \leq n-1$$
$$f(v_1) = f(v_1') - 1,$$
$$f(v_{1+i}) = f(v_1) - 2i; \ 1 \leq i \leq n-1$$

Case 2: n is odd.

$$f(u) = 2n-1, f(v) = 0, \ f(u') = 2n, \ f(v') = 1, f(u_1') = 2,$$
$$f(u_1') = n-1+i; 1 \leq i \leq n-1$$
$$f(u_1) = 2(i+1); 1 \leq i \leq \frac{n-3}{2}$$
$$f\left(u_{\frac{n-3}{2}+i}\right) = 5n+4-2i 1 \leq i \leq \frac{n+3}{2}$$
$$f(v_1') = 6n+4-i; 1 \leq i \leq n$$
$$f(v_1) = fv_n' - 1$$
$$f(v_{1+i}) = f(v_1) - 2i; 1 \leq i \leq n-1$$

Conclusion:

In this project graceful and graceful labeling of graphs. Mainly concentrated about the some basic definitions have been discussed. Also we have discussed about some theorems. The dissertation discusses that on strong vertex graceful graphs. We obtained necessary and sufficient condition for the edge graceful labeling of some trees. Next this dissertation discusses that graceful and odd graceful labeling of some trees.The result discussed in the dissertation may be used to stay about various graph theory invariants.

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