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SUM OF POWERS OF NATURAL NUMBERS USING POLYNOMIALS

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Abstract:

There are several ingenious methods to determine the sum of powers of natural numbers. In this paper, we define certain class of polynomials and examine their properties in detail. Further, we will see how the sum of powers of natural numbers could be easily evaluated by integrating such class of polynomials.

Key Words: Sum of k^{th} Powers of Natural Numbers, Polynomials, Bernoulli Numbers, Fundamental Theorem of Calculus.

1. Definition:

Let us denote the sum of kth powers of first n natural numbers by

$$S_k(n) = 1^k + 2^k + \cdots + n^k \tag{1}$$

We notice that, $S_0(n) = n$

In view of formulas presented in [1, 2], we know that $S_k(n)$ is a polynomial in n of degree $k + 1$.

2. Definition of Polynomials $Q_n(x)$

Let us define set of polynomials $Q_n(x)$ where $n \geq 0$ is an integer, recursively as follows:

$$\begin{aligned} Q_n(0) &= B_n \\ Q_0(x) &= 1 \\ Q_n'(x) &= n \times Q_{n-1}(x) \end{aligned} \tag{2}$$

Where B_n is the nth Bernoulli number.

3. Definition of Bernoulli Numbers:

Bernoulli Numbers are numbers which occur as coefficients of $\frac{x^n}{n!}$ in the Taylor's series expansion of $\frac{x}{e^x-1}$ about $x = 0$. We denote the nth Bernoulli Number by B_n. For knowing more about Bernoulli numbers and their properties see [2]

Thus by definition we get $$\frac{x}{e^x-1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \tag{3}$$

We notice that the constant term of $\frac{x}{e^x-1}$ is 1 and so we obtain $B_0 = 1$.

In view of [3], we know that the Bernoulli numbers satisfy the equation

$$\sum_{j=0}^{n} \binom{n+1}{j} B_j = 0 \tag{4}$$

Using the fact that $B_0 = 1$ and (4), the first few Bernoulli numbers are given by

$$\left.\begin{aligned} &B_0 = 1, B_1 = \pm\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}, B_5 = 0, B_6 = \frac{1}{42}, B_7 = 0, B_8 = -\frac{1}{30}, B_9 = 0 \\ &B_{10} = \frac{5}{66}, B_{11} = 0, B_{12} = -\frac{691}{2730}, B_{13} = 0, B_{14} = \frac{7}{6}, B_{15} = 0, B_{16} = -\frac{3617}{510}, \ldots \end{aligned}\right\} \tag{5}$$

4. Evaluation of $Q_n(x)$ *for* $n \geq 1$

Let us determine $Q_n(x)$ for $n = 1, 2, 3$ etc. By (2), we get $Q_0(x) = 1$.

Let $n = 1$. Then $Q_1'(x) = 1 \times Q_0(x) = 1$

Integrating with respect to x on both sides

$$Q_1(x) = x + c_1 \tag{6}$$

To find c_1, put $x = 0$ in (6), $Q_1(0) = c_1$

From (2), $Q_1(0) = B_1 = 1/2$
Thus, we get $c_1 = 1/2$ and using this in (6),

$$Q_1(x) = x + \frac{1}{2} \tag{7}$$

Let $n = 2$

$$Q_2'(x) = 2 \times Q_1(x) = 2x + 1$$

Integrating with respect to x on both sides

$$Q_2(x) = x^2 + x + c_2 \tag{8}$$

To find c_2, put $x = 0$ in (8), $Q_2(0) = c_2$
From (2), $Q_2(0) = B_2 = 1/6$
Thus, we get $c_2 = 1/6$ and using this in (8),

$$Q_2(x) = x^2 + x + \frac{1}{6} \tag{9}$$

Let $n = 3$

$$Q_3'(x) = 3 \times Q_2(x) = 3x^2 + 3x + \frac{1}{2}$$

Integrating with respect to x on both sides

$$Q_3(x) = x^3 + \frac{3}{2}x^2 + \frac{1}{2}x + c_3 \tag{10}$$

To find c_3, put $x = 0$ in (10)

$$Q_3(0) = c_3$$

From (2), $Q_3(0) = B_3 = 0$
Thus, we get $c_3 = 0$ and using this in (10),

$$Q_3(x) = x^3 + \frac{3}{2}x^2 + \frac{1}{2}x \tag{11}$$

Similarly we can compute $Q_n(x)$ for other values of n.

5. Integrals of $Q_n(x)$ for $n \geq 1$

Let us integrate $Q_n(x)$ for $n \geq 1$ with respect to x between any 2 consecutive natural numbers, say $k - 1$ and k.

Let us consider $n = 1$

$$\int_{k-1}^{k} Q_1(x)\,dx = \int_{k-1}^{k} \left(x + \frac{1}{2}\right) dx = k \tag{12}$$

Now considering $S_k(n)$ for $k = 1$ and using (12), we get

$$S_1(n) = 1 + 2 + 3 + \cdots + n$$
$$= \int_0^1 Q_1(x)\,dx + \int_1^2 Q_1(x)\,dx + \cdots + \int_{n-1}^{n} Q_1(x)\,dx$$
$$= \int_0^n Q_1(x)\,dx = \int_0^n \left(x + \frac{1}{2}\right) dx = \frac{n^2}{2} + \frac{n}{2} = \frac{n(n+1)}{2}$$

Thus, we have determined the value of $S_1(n)$ by integrating $Q_1(x)$
Let us consider $n = 2$

$$\int_{k-1}^{k} Q_2(x)\,dx = \int_{k-1}^{k} \left(x^2 + x + \frac{1}{6}\right) dx = k^2 \tag{13}$$

Now considering $S_k(n)$ for $k = 2$ and using (13), we get

$$S_2(n) = 1^2 + 2^2 + 3^2 + \cdots + n^2$$
$$= \int_0^1 Q_2(x)\,dx + \int_1^2 Q_2(x)\,dx + \ldots + \int_{n-1}^{n} Q_2(x)\,dx$$
$$= \int_0^n Q_2(x)\,dx = \int_0^n \left(x^2 + x + \frac{1}{6}\right) dx = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} = \frac{n(n+1)(2n+1)}{6}$$

Thus, we have determined the value of $S_2(n)$ by integrating $Q_2(x)$
Now we see if this pattern can be generalized to evaluate $S_k(n)$ for any given k by integrating $Q_k(x)$. In order to do this, first we will prove the following theorem.

6. Theorem 1:

If $Q_n(x)$ are the polynomials as defined in (2), then $Q_n(k) - Q_n(k-1) = n.k^{n-1}$ (14)

Proof:

We will prove (14) using mathematical induction on n.

Let $n = 1$. Then $Q_1(k) - Q_1(k-1) = k + \frac{1}{2} - \left(k - 1 + \frac{1}{2}\right) = 1 = 1.k^0$. Thus (14) is true if $n = 1$. Now consider $n = 2$. Then we obtain

$$Q_2(k) - Q_2(k-1) = k^2 + k + \frac{1}{6} - \left((k-1)^2 + (k-1) + \frac{1}{6}\right) = 2k = 2.k^{2-1}$$

Hence (14) is true for $n = 2$ also. So, by Induction Hypothesis, we assume that (14) is true for all values of $n = 1, 2,, . . . , m$. We will show that it is also true for $n = m + 1$.

$$(m+1).k^m = m.k^{m-1}.k + k^m$$
$$= [Q_m(k) - Q_m(k-1)]\,k + k^m \quad \text{(Using Induction Hypothesis for } n = m\text{)}$$

Now using (2) and re-arranging the terms, we obtain

$$(m+1).k^m - k^m = \frac{k}{(m+1)} Q'_{m+1}(k) - \frac{k}{(m+1)} Q'_{m+1}(k-1)$$
$$mk^{m-1} = \frac{1}{(m+1)} Q'_{m+1}(k) - \frac{1}{(m+1)} Q'_{m+1}(k-1)$$

Using Fundamental Theorem of Calculus, integrating above equation on both sides with respect to k, we get $k^m = \frac{1}{(m+1)}[Q_{m+1}(k) - Q_{m+1}(k-1)] + \lambda$, where λ is constant of integration.

If $m = 0$, then from above equation, we obtain $1 = [Q_1(k) - Q_1(k-1)] + \lambda$. Now using (7), we get $1 = \left(k + \frac{1}{2}\right) - \left(k - 1 + \frac{1}{2}\right) + \lambda$ from which $\lambda = 0$.

Hence, $(m+1)k^m = Q_{m+1}(k) - Q_{m+1}(k-1)$ proving (14) for $n = m + 1$ also.

So, by principle of mathematical induction, (14) is true for all non - negative integers n.

This completes the proof.

We now establish the following important theorem.

7. Theorem 2:

If $k \geq 0$ and $n \geq 1$ are integers then $S_k(n) = \int_0^n Q_k(x)\, dx$ (15)

Proof:

Using (2) and (14), we see that

$$\int_{k-1}^{k} Q_n(x)\, dx = \int_{k-1}^{k} \frac{1}{(n+1)} Q'_{n+1}(x) dx = \frac{1}{(n+1)}[Q_{n+1}(k) - Q_{n+1}(k-1)]$$
$$= \frac{1}{(n+1)} \times (n+1)k^n = k^n \qquad (16)$$

Hence using additive property of integrals and (16), we obtain

$$S_k(n) = 1^k + 2^k + 3^k + \cdots + n^k$$
$$= \int_0^1 Q_k(x)\, dx + \int_1^2 Q_k(x)\, dx + \ldots + \int_{n-1}^{n} Q_k(x)\, dx$$
$$= \int_0^n Q_k(x)\, dx$$

This completes the proof.

8. Conclusion:

In this paper, we have defined the polynomials $Q_n(x)$ for any natural number n as in (2). In fact, these polynomials are translated form of the well known Bernoulli polynomials in mathematics literature. Hence using the modified form of Bernoulli polynomials, we have first proved (14) in theorem 1 and using this result as well as the well known theorems in calculus, we have established a nice formula that sum of kth powers of first n natural numbers namely $S_k(n)$ is obtained by integrating $Q_k(x)$ over the closed and bounded interval $[0, n]$ for fixed natural number n as in (15) of theorem 2. Hence knowing the modified form of Bernoulli polynomials, one can immediately obtain $S_k(n)$ for given k.

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