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INVESTIGATION OF PERFECT SQUARES AMONG NARAYANA NUMBERS

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Abstract:

Narayana numbers named after the Indian mathematician T. V. Narayana (1930–1987) plays a vital role in combinatorial problems. In this paper, we discuss about for what choices of a and b the Narayana numbers $N(a,b)$forms a perfect square? In view of answering this question, three cases were discussed by considering odd, even and some specific positive integers. Using combinatorial arguments, divisibility properties, binomial coefficients, we have established three theorems in this paper.

Key Words: Narayana Numbers, Binomial coefficients, Catalan Numbers, Divisibility Property

1. Introduction:

Narayana numbers are a sequence of positive integers that arise in various mathematical contexts, including combinatorics, algebra, and geometry. The Narayana numbers are denoted by $N(n,k), n\epsilon N^{+}, 1 \leq k \leq n$ and forms a triangular array of natural numbers, called Narayana triangle that occurs in various counting problems, which is a fundamental problem in combinatorial geometry. Solution for Dyck words, a counting problem where number of words containing $n$pairs of parentheses, which are correctly matched and which contain k distinct nesting can be given in terms of Narayana numbers $N(n,k)$. Narayana numbers have several interesting properties, such as recurrence relations, generating functions, and algebraic identities. R. Sivaraman discussed two interesting results on Catalan numbers [5] and combinatorial identities of binomial coefficients and its properties associated withNarayana numbers [6]. These papers provides good insight in to understanding of Narayana numbers. A perfect square is a non-negative integer that can be expressed as the product of an integer with itself. This paper provides three conditions for which Narayana numbers can also be perfect squares. Detailed proofs have been presented.

2. Definitions:

2.1 Greatest Common Divisor:

Let a and b be positive integers. A number d is said to be a greatest common divisor of a, b if

- The number d divides both a and b
- If there is a number c which divides both a and b then c divides$d$

The greatest common divisor of a and b is denoted by (a,b)

2.2 Binomial Coefficient:

The numbers formed through two integers n and r where $n \geq 0, 0 \leq r \leq n$ given by the expression $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ are called binomial coefficients. Also, in binomial coefficients$\binom{n}{r} = \binom{n}{n-r} = \frac{n}{r}\binom{n-1}{r-1}$. Binomial coefficients form the coefficients of the binomial expansion $(a+b)^n$ for $n \geq 0$.

2.3 Narayana Numbers:

The Narayana numbers are positive integers defined by

$$N(a,b) = \frac{1}{a} \times \binom{a}{b} \times \binom{a}{b-1}, \text{ where } 1 \leq b \leq a+1.$$

2.4 Catalan Numbers:

The nth Catalan number is given by the expression $C_n = \frac{1}{n+1}\binom{2n}{n}$

3. Some Results on Perfect Squares of Narayana Numbers:

3.1 Theorem 1:

The Narayana numbers $N\left(n^2, \frac{n^2+1}{2}\right)$ is a perfect square if n is odd

Proof:

By definition, Narayana numbers are given by

$$N(a,b) = \frac{1}{a} \times \binom{a}{b} \times \binom{a}{b-1}$$

$$= \frac{b}{a} \times \binom{a}{b} \times \frac{1}{b} \times \binom{a}{b-1}$$

$$= \frac{b}{a} \times \binom{a}{b} \times \frac{1}{b} \times \frac{a!}{(b-1)!(a-b+1)!} = \frac{b}{a} \times \binom{a}{b} \times \frac{a!}{b(b-1)!\,(a-b+1)!}$$

$$= \frac{b}{a} \times \binom{a}{b} \times \frac{a!}{b!\ (a-b+1)(a-b)!} = \frac{b}{a} \times \binom{a}{b} \times \frac{1}{(a-b+1)} \times \frac{a!}{b!\,(a-b)!}$$

$$= \frac{b}{a} \times \binom{a}{b} \times \frac{1}{(a-b+1)} \times \binom{a}{b}$$

$$= \frac{b}{a(a-b+1)} \times \binom{a}{b}^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1)}$$

In view of (1), it is enough to prove $N(a,b)$ is a perfect square, in above it is enough to show that $\frac{b}{a(a-b+1)}$ is a square

If n is an odd integer then n^2 is also an odd integer

Hence,$n^2 + 1$ will be an even integer which implies $\frac{n^2+1}{2}$ is an integer

So consider, $a = n^2, b = \frac{n^2+1}{2}$

Hence, $\frac{b}{a(a-b+1)} = \frac{\frac{n^2+1}{2}}{n^2\left(n^2-\frac{n^2+1}{2}+1\right)} = \frac{\frac{n^2+1}{2}}{n^2\left(\frac{n^2+1}{2}\right)} = \frac{1}{n^2}$

Now using (1), we have

$$N\left(n^2, \frac{n^2+1}{2}\right) = \frac{1}{n^2} \times \binom{n^2}{\frac{n^2+1}{2}}^2 = \left(\frac{1}{n}\binom{n^2}{\frac{n^2+1}{2}}\right)^2 \ldots\ldots\ldots\ldots\ldots \text{(2)}$$

In view of (2), for proving $N\left(n^2, \frac{n^2+1}{2}\right)$ is a perfect square when n is odd, we now need to prove that$n$divides $\binom{n^2}{\frac{n^2+1}{2}}$.

Now, $\left(n^2, \frac{n^2+1}{2}\right) = \left(\frac{n^2-1}{2}, \frac{n^2+1}{2}\right) = \left(\frac{n^2-1}{2}, 1\right) = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(3)}$

We know that $\frac{n}{(n,r)} | \binom{n}{r}$

Therefore $\frac{n^2}{\left(n^2, \frac{n^2+1}{2}\right)} \mid \binom{n^2}{\frac{n^2+1}{2}} = \frac{n^2}{1} \mid \binom{n^2}{\frac{n^2+1}{2}} = n^2 \mid \binom{n^2}{\frac{n^2+1}{2}}$

$$\Longrightarrow n^2 \text{ divides } \binom{n^2}{\frac{n^2+1}{2}} \Longrightarrow n \text{ divides } \binom{n^2}{\frac{n^2+1}{2}}$$

Therefore, $N\left(n^2, \frac{n^2+1}{2}\right)$is a perfect square if n is odd.

This completes the proof.

3.2 Theorem 2:

The Narayana numbers $N\left(n^2 - 2, \frac{n^2-2}{2}\right)$ is a perfect square if n is even

Proof:

From (1),$N\left(n^2 - 2, \frac{n^2-2}{2}\right) = \frac{\frac{n^2-2}{2}}{(n^2-2)\times(n^2-2-\frac{n^2-2}{2}+1)} \times \binom{n^2-2}{\frac{n^2-2}{2}}^2 = \frac{\frac{n^2-2}{2}}{(n^2-2)(\frac{n^2}{2})} \times \binom{n^2-2}{\frac{n^2-2}{2}}^2$

$$= \frac{\mathbf{1}}{\mathbf{n^2}} \times \binom{n^2-2}{\frac{n^2-2}{2}}^2 = \left[\frac{1}{n} \times \binom{n^2-2}{\frac{n^2-2}{2}}\right]^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(4)}$$

Hence by (4), in order to prove that$N\left(n^2 - 2, \frac{n^2-2}{2}\right)$ is a perfect square if n is even, it is enough toprove n divides $\binom{n^2-2}{\frac{n^2-2}{2}}$.

If n is even, then $n^2 - 2$ is even and so $\frac{n^2-2}{2}$ is an integer.

Since n is even, $n = 2\lambda, \lambda \in Z^+$

$$\therefore \frac{n^2}{2} = \frac{4\lambda^2}{2} = 2\lambda^2 = 2 \times \lambda \times \lambda = n\lambda$$

$$\therefore \frac{n^2}{2} = n\lambda \Rightarrow n \mid \frac{n^2}{2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(5)}$$

$\frac{n^2-2}{2} + 1 \left| \binom{n^2-2}{\frac{n^2-2}{2}} \Rightarrow \frac{n^2}{2} \right| \binom{n^2-2}{\frac{n^2-2}{2}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ (6) (By Definition 2.4)

From (5) and (6), n divides $\binom{n^2-2}{\frac{n^2-2}{2}}$.

Hence $N\left(n^2 - 2, \frac{n^2-2}{2}\right)$ is a perfect square if n is even

3.3 Theorem 3:

The Narayana numbers $N\left(n^2(n^2+1), (n^2+1)\right)$ is a perfect square, if$n$ is any positive integer

Proof:

From (1), $N\left(n^2(n^2+1),(n^2+1)\right) = \frac{(n^2+1)}{n^2(n^2+1)\times(n^2(n^2+1)-(n^2+1)+1)} \times \binom{n^2(n^2+1)}{n^2+1}^2$

$$= \frac{(n^2+1)}{n^2(n^2+1)\times(n^4+n^2-n^2-1+1)} \times \binom{n^2(n^2+1)}{n^2+1}^2$$

$$= \frac{(n^2+1)}{n^2(n^2+1)n^4} \times \binom{n^2(n^2+1)}{n^2+1}^2 = \frac{1}{n^6} \times \binom{n^2(n^2+1)}{n^2+1}^2$$

$$= \left[\frac{1}{n^3} \times \binom{n^2(n^2+1)}{n^2+1}\right]^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

In view of (7), to prove $N\left(n^2(n^2+1),(n^2+1)\right)$ is a perfect square if n is any positive integer,it is enough to prove n^3 divides $\binom{n^2(n^2+1)}{n^2+1}$(i.e.) $\frac{\binom{n^2(n^2+1)}{n^2+1}}{n^3}$ is an integer

We know that, $\binom{n}{r} = \frac{n}{r}\binom{n-1}{r-1}$

$$\binom{n^2(n^2+1)}{n^2+1} = \frac{n^2(n^2+1)}{n^2+1}\binom{n^2(n^2+1)-1}{n^2+1-1} = n^2\binom{n^4+n^2-1}{n^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

From (7), we need to prove n^3 divides $\binom{n^2(n^2+1)}{n^2+1}$. Hence using (8), we have

$$\text{(i.e.)}\, n^3 \mid \binom{n^2(n^2+1)}{n^2+1} \Rightarrow n^3 \mid n^2\binom{n^4+n^2-1}{n^2} \Rightarrow n \mid \binom{n^4+n^2-1}{n^2}$$

That is to prove $n^3 \mid \binom{n^2(n^2+1)}{n^2+1}$it is enough to prove $n \mid \binom{n^4+n^2-1}{n^2}$which in turn implies$n^3 \mid \binom{n^2(n^2+1)}{n^2+1}$

By definition of binomial coefficient,

$$\binom{n}{r} = \frac{n(n-1)(n-2)\ldots\ldots\ldots\ldots(n-(r-1))}{1.2.3\ldots\ldots\ldots\ldots\ldots\ldots\ldots r}$$

$$\text{Hence,}\binom{n^4+n^2-1}{n^2} = \frac{(n^4+n^2-1)(n^4+n^2-2)(n^4+n^2-3)\ldots\ldots\ldots(n^4+n^2-1-(n^2-1))}{1.2.3\ldots\ldots\ldots\ldots(n^2-1)n^2}$$

$$\Rightarrow \binom{n^4+n^2-1}{n^2} = \frac{(n^4+n^2-1)(n^4+n^2-2)(n^4+n^2-3)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(n^4)}{1.2.3\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(n^2-1)n^2}$$

$$\Rightarrow n^2 \text{ is the factor of } \binom{n^4+n^2-1}{n^2} \Rightarrow n \text{divides} \binom{n^4+n^2-1}{n^2} \qquad \text{(From 8)}$$

Hence, n^3 divides $\binom{n^2(n^2+1)}{n^2+1}$.

Therefore, $N\left(n^2(n^2+1),(n^2+1)\right)$ is a perfect square if n is a positive integer.

This completes the proof.

4. Conclusion:

Among Narayana numbers, which kind of values of a, b provides perfect squares? This question was briefly discussed in this paper through three theorems.In three theorems discussed in section 3, we always get perfect squares. But these three cases does not provide the complete set of perfect squares that can possibly exist among Narayana numbers. For example, N (1728,28) is a perfect square, but this doesn't come under any of three categories discussed in the paper. In future study, we will consider determining such perfect squares. There are several possibilities of extending or modifying the results proved in this paper.

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