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LUCAS NUMBERS TRIANGLE

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Abstract:

Among several number triangles that exist in mathematics, Pascal's triangle is well known for its extra-ordinary combinatorial properties and applications. It is well known that stacking entries of Pascal's triangle in a particular fashion and adding them in North – East diagonal direction, we can generate Fibonacci numbers. In this paper, I will introduce a number triangle constructed similar to Pascal's triangle through which I have proved eight interesting and new mathematical properties and also have generated Lucas numbers which is connected with Golden Ratio.

Key Words: Binomial Coefficients, Slant Diagonals, Square Pyramidal Numbers, Hockey Stick Property, Lucas Numbers.

1. Introduction:

French mathematician Edward Lucas, is well known for creating several amusing problems and concepts in mathematics including that of Tower of Hanoi puzzle. The sequence beginning with 2, 1 and other terms are obtained by adding the two previous terms is called Lucas Sequence, named in honor of Lucas. This sequence possesses almost all properties similar to Fibonacci sequence, since it has same generation rule. In this paper, I will introduce a simple number triangle similar to Pascal's triangle and prove some mathematical properties related to its entries and generate Lucas Numbers.

2. Construction of Number Triangle:

In this section, I will define rules to construct a number triangle whose properties will be discussed extensively in this paper. Let $t_{n,m}$ be the number located in nth row, mth position in the number triangle where $1 \le m \le n$. We now define the numbers of the triangle in the following way:

For any natural number n, let $t_{1,1} = 2, t_{n,1} = 1$ and for $2 \le m \le n$ we define $t_{n,m} = \binom{n-1}{m-1} + \binom{n-2}{m-2}$ (2.1). We notice that $\binom{n}{r}$ is the binomial coefficient and from (2.1), $t_{n,n} = 2$. With these conditions the number triangle is shown in Figure 1, which we call as Lucas number triangle.

Row Number	
1	2
2	1 2
3	1 3 2
4	1 4 5 2
5	1 5 9 7 2
6	1 6 14 16 9 2
7	1 7 20 30 25 11 2
8	1 8 27 50 55 36 13 2
9	1 9 35 77 105 91 49 15 2
10	1 10 44 112 182 196 140 64 17 2

. . . .

Figure 1: Lucas Number Triangle

Except first row, we notice that for $n \ge 2$ the numbers in the Lucas number triangle are coefficients in the expansion $(a+b)^{n-2}(a+2b)$ (2.2)

3. Properties of Lucas Number Triangle:

3.1 Theorem 1 (Row Sum Property):

If $n \ge 2$, the sum of numbers in nth row of Lucas number triangle is given by $\sum_{m=1}^{n} t_{n,m} = 3 \times 2^{n-2}$ (3.1)

Proof:

Using (2.1) and the sum of binomial coefficients identity $\sum_{k=0}^{n}\binom{n}{k}=2^n$ we have

$$\sum_{m=1}^{n} t_{n,m} = t_{n,1}+t_{n,2}+t_{n,3}+\cdots+t_{n,n-1}+t_{n,n}$$

$$=1+\sum_{m=2}^{n}\left[\binom{n-1}{m-1}+\binom{n-2}{m-2}\right]$$

$$=1+\left[\binom{n-1}{1}+\binom{n-2}{0}\right]+\left[\binom{n-1}{2}+\binom{n-2}{1}\right]+\cdots+\left[\binom{n-1}{n-1}+\binom{n-2}{n-2}\right]$$

$$=\left[\binom{n-1}{0}+\binom{n-1}{1}+\binom{n-1}{2}+\cdots+\binom{n-1}{n-1}\right]+\left[\binom{n-2}{0}+\binom{n-2}{1}+\binom{n-2}{2}+\cdots+\binom{n-2}{n-2}\right]$$

$$=2^{n-1}+2^{n-2}=2^{n-2}\times 3$$

Note that we can also prove (3.1) using (2.2). In this case for $n\geq 2$ the row sum can be obtained by just taking a = 1, b = 1 giving $(1+1)^{n-2}\times(1+2)=2^{n-2}\times 3$.

This completes the proof.

3.2 Theorem 2 (Alternate Sum Property):

If $n\geq 3$, the alternate sum of nth row entries of Lucas number triangle is zero. That is, $\sum_{m=1}^{n}(-1)^{m-1}t_{n,m}=0$ (3.2)

Proof:

$$\sum_{m=1}^{n}(-1)^{m-1}t_{n,m}=t_{n,1}-t_{n,2}+t_{n,3}-t_{n,4}+\cdots+(-1)^{n-1}t_{n,n}$$

Using (2.1), we get $\sum_{m=1}^{n}(-1)^{m-1}t_{n,m}=1+\sum_{m=2}^{n}(-1)^{m-1}\left[\binom{n-1}{m-1}+\binom{n-2}{m-2}\right]$

If n is odd say, n = 2r + 1, for r = 1, 2, 3, 4, 5, . . ., we get

$$\sum_{m=1}^{2r+1}(-1)^{m-1}t_{2r+1,m}=1+\sum_{m=2}^{2r+1}(-1)^{m-1}\left[\binom{2r}{m-1}+\binom{2r-1}{m-2}\right]$$

$$=1-\left[\binom{2r}{1}+\binom{2r-1}{0}\right]+\left[\binom{2r}{2}+\binom{2r-1}{1}\right]-\cdots-\left[\binom{2r}{2r-1}+\binom{2r-1}{2r-2}\right]+\left[\binom{2r}{2r}+\binom{2r-1}{2r-1}\right]$$

Writing $1=\binom{2r}{0}$ and the fact that $\sum_{k=0}^{n}(-1)^k\binom{n}{k}=0$ we get

$$\sum_{m=1}^{2r+1}(-1)^{m-1}t_{2r+1,m}=\sum_{k=0}^{2r}(-1)^k\binom{2r}{k}-\sum_{k=0}^{2r-1}(-1)^k\binom{2r-1}{k}=0-0=0$$

Similarly, if n is even say, n = 2r + 2, for r = 1, 2, 3, 4, 5, . . ., we get

$$\sum_{m=1}^{2r+2}(-1)^{m-1}t_{2r+2,m}=1+\sum_{m=2}^{2r+2}(-1)^{m-1}\left[\binom{2r+1}{m-1}+\binom{2r}{m-2}\right]$$

$$=1-\left[\binom{2r+1}{1}+\binom{2r}{0}\right]+\left[\binom{2r+1}{2}+\binom{2r}{1}\right]-\cdots+\left[\binom{2r+1}{2r}+\binom{2r}{2r-1}\right]-\left[\binom{2r+1}{2r+1}+\binom{2r}{2r}\right]$$

Writing $1=\binom{2r+1}{0}$ and the fact that $\sum_{k=0}^{n}(-1)^k\binom{n}{k}=0$ we get

$$\sum_{m=1}^{2r+2}(-1)^{m-1}t_{2r+2,m}=\sum_{k=0}^{2r+1}(-1)^k\binom{2r+1}{k}-\sum_{k=0}^{2r}(-1)^k\binom{2r}{k}=0-0=0$$

This proves (3.2) and hence completes the proof.

Note that we can also prove (3.2) using (2.2). In this case for $n\geq 3$ the alternating sum in each row can be obtained by just taking a = 1, b = –1 giving $(1-1)^{n-2}\times(1-2)=0$.

3.3 Theorem 3:

The second South – East slant diagonal entries of Lucas number triangle are odd numbers. That is, for $n \geq 2$, we have $t_{n,n-1} = 2n-3 \quad (3.3)$

Proof:

Using (2.1), we have

$$t_{n,n-1} = \binom{n-1}{n-2} + \binom{n-2}{n-3} = \binom{n-1}{1} + \binom{n-2}{1} = (n-1)+(n-2) = 2n-3$$

For $n \geq 2$, we see that $2n - 3$ generates all odd natural numbers. In fact, these numbers are shown in Orange color in Figure 1. This completes the proof.

3.4 Theorem 4:

The third South – East slant diagonal entries of Lucas number triangle are square numbers. That is, for $n \geq 3$, we have $t_{n,n-2} = (n-2)^2 \quad (3.4)$

Proof:

Using (2.1), we have

$$t_{n,n-2} = \binom{n-1}{n-3} + \binom{n-2}{n-4} = \binom{n-1}{2} + \binom{n-2}{2} = \frac{(n-1)(n-2)}{2} + \frac{(n-2)(n-3)}{2} = (n-2)^2$$

For $n \geq 3$, we see that $(n-2)^2$ generates all perfect squares. In fact, these numbers are shown in Green color in Figure 1. This completes the proof.

3.5 Theorem 5:

The fourth South – East slant diagonal entries of Lucas number triangle are square pyramidal numbers. That is, for $n \geq 4$, we have

$$t_{n,n-3} = 1^2 + 2^2 + 3^2 + \cdots + (n-3)^2 = \frac{(n-2)(n-3)(2n-5)}{6} \quad (3.5)$$

Proof:

Using (2.1), we have

$$t_{n,n-3} = \binom{n-1}{n-4} + \binom{n-2}{n-5} = \binom{n-1}{3} + \binom{n-2}{3} = \frac{(n-1)(n-2)(n-3)}{6} + \frac{(n-2)(n-3)(n-4)}{6}$$

$$= \frac{(n-3)(n-2)(2n-5)}{6} = 1^2 + 2^2 + 3^2 \cdots + (n-3)^2$$

For $n \geq 4$, we see that $1^2 + 2^2 + 3^2 + \cdots + (n-3)^2$ generates all square pyramidal numbers which are sum of consecutive square numbers. In fact, these numbers are shown in Pink color in Figure 1. This completes the proof.

4. Hockey Stick Properties:

In this section I will present three properties known as Hockey – Stick properties analogous to the properties satisfied by binomial coefficients in Pascal's triangle. First, I will present a version of Hockey – Stick formula (for proof see [3]) satisfied for entries in Pascal's triangle.

$$\sum_{r=0}^{q} \binom{p+r}{r} = \binom{p+q+1}{q} \quad (4.1)$$

4.1 Theorem 6:

For any natural number *n*, the South – East outer diagonal diagonal entries of Lucas number triangle satisfy the Hockey Stick Property given by $\sum_{u=1}^{n} t_{u,u} = t_{n+1,n} + 1 \quad (4.2)$

Proof:

From (2.1), we see that for any natural number *n*, $t_{n,n} = 2$. Hence $\sum_{u=1}^{n} t_{u,u} = \sum_{u=1}^{n} 2 = 2n$.

Similarly using (2.1), we have

$$t_{n+1,n} + 1 = \binom{n}{n-1} + \binom{n-1}{n-2} + 1 = \binom{n}{1} + \binom{n-1}{1} + 1 = n + (n-1) + 1 = 2n$$

This proves (4.2) and completes the proof.

4.2 Theorem 7:

For any natural number $n \geq 2$, the South – East slant diagonal entries of Lucas number triangle satisfy the Hockey Stick Property given by $\sum_{u=0}^{k} t_{n+u,u+1} = t_{n+k+1,k+1}$ (4.3)

Proof:

Using (2.1) and (4.1), we have

$$\sum_{u=0}^{k} t_{n+u,u+1} = t_{n,1} + t_{n+1,2} + t_{n+2,3} + \cdots + t_{n+k-1,k} + t_{n+k,k+1}$$

$$= 1 + \left[\binom{n}{1} + \binom{n-1}{0}\right] + \left[\binom{n+1}{2} + \binom{n}{1}\right] + \cdots + \left[\binom{n+k-2}{k-1} + \binom{n+k-3}{k-2}\right] + \left[\binom{n+k-1}{k} + \binom{n+k-2}{k-1}\right]$$

$$= \left[\binom{n-1}{0} + \binom{n}{1} + \binom{n+1}{2} + \cdots + \binom{n+k-2}{k-1} + \binom{n+k-1}{k}\right] +$$

$$\left[\binom{n-1}{0} + \binom{n}{1} + \binom{n+1}{2} + \cdots + \binom{n+k-3}{k-2} + \binom{n+k-2}{k-1}\right]$$

$$= \sum_{r=0}^{k} \binom{n-1+r}{r} + \sum_{r=0}^{k-1} \binom{n-1+r}{r} = \binom{n-1+k+1}{k} + \binom{n-1+k-1+1}{k-1}$$

$$= \binom{n+k}{k} + \binom{n+k-1}{k-1} = t_{n+k+1,k+1}$$

This proves (4.3) and completes the proof.

4.3 Theorem 8:

For any natural number n, the South – West slant diagonal entries of Lucas number triangle satisfy the Hockey Stick Property given by $\sum_{u=0}^{k} t_{n+u,n} = t_{n+k+1,n+1}$ (4.4)

Proof:

Using (2.1) and (4.1), we have

$$\sum_{u=0}^{k} t_{n+u,n} = t_{n,n} + t_{n+1,n} + t_{n+2,n} + \cdots + t_{n+k-1,n} + t_{n+k,n}$$

$$= \left[\binom{n-1}{n-1} + \binom{n-2}{n-2}\right] + \left[\binom{n}{n-1} + \binom{n-1}{n-2}\right] + \left[\binom{n+1}{n-1} + \binom{n}{n-2}\right] + \cdots + \left[\binom{n+k-1}{n-1} + \binom{n+k-2}{n-2}\right]$$

$$= \left[\binom{n-1}{0} + \binom{n}{1} + \binom{n+1}{2} + \cdots + \binom{n+k-2}{k-1} + \binom{n+k-1}{k}\right] +$$

$$\left[\binom{n-2}{0} + \binom{n-1}{1} + \binom{n}{2} + \cdots + \binom{n+k-3}{k-1} + \binom{n+k-2}{k}\right]$$

$$= \sum_{r=0}^{k} \binom{n-1+r}{r} + \sum_{r=0}^{k} \binom{n-2+r}{r} = \binom{n-1+k+1}{k} + \binom{n-2+k+1}{k}$$

$$= \binom{n+k}{k} + \binom{n+k-1}{k} = \binom{n+k}{n} + \binom{n+k-1}{n-1} = t_{n+k+1,n+1}$$

This proves (4.4) and completes the proof.

5. Generating Lucas Numbers:

In this section, I will present ways to generate Lucas numbers from the number triangle in Figure 1, whose entries are defined as in (2.1). First, we will arrange the numbers of Lucas number triangle of Figure 1 aligned in vertical column as shown below.

Row Number										
1	2									
2	1	2								
3	1	3	2							
4	1	4	5	2						
5	1	5	9	7	2					
6	1	6	14	16	9	2				
7	1	7	20	30	25	11	2			
8	1	8	27	50	55	36	13	2		
9	1	9	35	77	105	91	49	15	2	
10	1	10	44	112	182	196	140	64	17	2
	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .

Figure 2: Triangle Generating Lucas Numbers

If we add numbers in respective colors along North – East direction in Figure 2, we get 2, 1, 1 + 2 = 3, 1 + 3 = 4, 1 + 4 + 2 = 7, 1 + 5 + 5 = 11, 1 + 6 + 9 + 2 = 18, 1 + 7 + 14 + 7 = 29, . . .

Thus we get a sequence whose terms are given by 2, 1, 3, 4, 7, 11, 18, 29, 47, 76, . . . (5.1).

The sequence described in (5.1) is called the Lucas Sequence named after Edward Lucas. This sequence is analogous to Fibonacci Sequence except the fact that the first two terms in Lucas sequence are 2, 1 where as in Fibonacci sequence it is 1, 1. We notice that but for the first two terms in Lucas sequence, each term is sum of two previous two terms. We know that (see [7], [8]) the nth term of the Lucas sequence is given by $L_n = \varphi^n + (1-\varphi)^n$ (5.2) where $\varphi = \frac{1+\sqrt{5}}{2}$ is the Golden Ratio and $n \geq 0$.

Since $-1 < \frac{1-\varphi}{\varphi} < 0$ it follows that $\left(\frac{1-\varphi}{\varphi}\right)^n \to 0$ as $n \to \infty$. Hence, from (5.2), we have

$$\lim_{n\to\infty} \frac{L_{n+1}}{L_n} = \lim_{n\to\infty} \frac{\varphi^{n+1} + (1-\varphi)^{n+1}}{\varphi^n + (1-\varphi)^n} = \lim_{n\to\infty} \frac{\varphi + (1-\varphi)\left(\frac{1-\varphi}{\varphi}\right)^n}{1 + \left(\frac{1-\varphi}{\varphi}\right)^n} = \varphi \quad (5.3)$$

Thus from (5.3), we see that the ratio of successive Lucas numbers tends to Golden Ratio.

6. Conclusion:

By introducing a simple number triangle as in Figure 1, in this paper, I had proved eight interesting results in which theorems 6, 7 and 8 corresponding to Hockey Stick Properties in section 4 were new. Also, I have proved five interesting results in theorems 1 to 5 in section 3. These results will convey the beauty and explore the properties of the number triangle considered in this paper. In section 5, I had generated Lucas numbers through the number triangle considered and hence named it as Lucas Number Triangle. Through equation (5.3), I had shown a well-known property that the ratio of successive terms of Lucas sequence is the Golden Ratio. Thus, the number triangle introduced in Figure 1, is intimately connected with Lucas sequence which in turn is connected with Golden Ratio. In mathematics literature a similar triangle by the name (2,1) – Pascal's triangle exists, but the triangle introduced in this paper is different in several aspects from (2,1) – Pascal's triangle. In fact, I had proved the first five theorems using the definition of general term in combinatorial way as in (2.1). The results derived in this paper will provide ample scope foe proving further new results.

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