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# RADIO NUMBER OF SOME AVL TREES

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**Abstract:**

The Frequency Assignment Problem (FAP) can be modeled as a coloring problem on the graph where vertices represent transmitters and edges indicate closeness of the transmitters. A radio labelling of graphs is a variation of FAP. For a simple connected graph $\mathrm{G} = (\mathrm{V(G)}, \mathrm{E(G)})$, a radio labelling is a mapping $\mathrm{f}: \mathrm{V(G)} \to \{0,1,2,...\}$ such that $|\mathrm{f(u)} - \mathrm{f(v)}| \geq \mathrm{diam(G)} + 1 - \mathrm{d(u,v)}$ for each pair of distinct vertices $\mathrm{u}, \mathrm{v} \in \mathrm{V(G)}$, where $\mathrm{diam(G)}$ is the diameter of G and $\mathrm{d(u,v)}$ is the distance between u and v. The span of a radio labelling f, denoted by $\mathrm{span(f)}$, is the largest integer assigned to a vertex of G. The radio number $\mathrm{rn(G)}$ of G is the number $\min\{\mathrm{span(f)}\}$, where the minimum is taken over all possible radio labelling of G.In this article, we give a lower bound for radio number of any trees in terms of number of vertices, diameter and the weight of maximum Hamiltonian path. We also determine radio number of some AVL trees by using this bound.

**Key Words:** Channel assignment problem, Radio labelling, Radio number, Span.

## 1. Introduction:

Frequency Assignment Problem (FAP) occur in many different types of wireless communication networks. Over the past years, the rapid development of new wireless services like radio and TV broadcasting, mobile telephony, military communication, satellite communication resulted in a run out of the most important resource, frequencies in the radio spectrum. Rapid growth in the use of wireless communication services effect on scarcity and economic cost of available frequencies. However, reuse of frequencies can offer considerable economies but also leads to loss of quality of communication links. The use of (almost) the same frequency for multiple wireless connections can cause an interference between the signals that is unacceptable. This introduced the need for operators to develop frequency plans. The FAP is to assign frequencies to the transmitters in a network in a way which avoids interference and uses the spectrum as efficiently as possible. In 1980, Hale [11] has modeled FAP as a Graph labelling problem (in particular as a generalized graph coloring problem) and is an active area of research now. In a graph model for FAP, the transmitters are represented by the vertices of a graph and two vertices are adjacent or at distance two apart in the graph if the corresponding transmitters are very close or close to each other. Griggs and Yeh [10] concentrated on the fundamental case of $L(2,1)$-labellings. The $L(p,q)$-labelling problem $(p, q > 0)$ and its variants have been studied extensively; see $[2, 3, 8, 9, 10, 11, 12, 13, 28, 29]$. A major concern of this problem is to seek an assignment of labels (which are nonnegative integers) to the vertices of a graph (transmitters) such that the span (difference) between the largest and smallest labels used is minimized, subject to that adjacent vertices (very close transmitters) receive labels with separation at least $p$ and vertices at distance two apart (close transmitters) receive labels with separation at least $q$. Practically it has been observed that interference among transmitters may go beyond two levels. Extending the level of interference at largest possible (diameter of interference graph), in $2005$, a new model, namely the radio labelling problem was introduced by Chartrand et al. $[4, 5]$ and studied further in $15, 16, 27$. For a simple connected graph $G = (V(G), E(G))$, a radio labelling is a mapping $f: V(G) \to \{0,1,2,...\}$ such that

$$|f(u) - f(v)| \geq diam(G) + 1 - d(u,v)$$

for each pair of distinct vertices $u, v \in V(G)$, where $diam(G)$ is the diameter of $G$ and $d(u,v)$ is the distance between $u$ and $v$. The span of, denoted by $span(f)$, is defined as $\max_{u,v \in V(G)} |f(u) - f(v)|$. The radio number $rn(G)$ of $G$ is the number $\min\{span(f)\}$, where the minimum is taken over all possible radio labelling of $G$. A radio labelling $f$ of $G$ is called minimal if $span(f) = rn(G)$. Without loss of generality, for a minimal radio labelling $f$ we assume that $\min_{v \in V(G)} f(v) = 0$, otherwise the span of $f$ can be reduced further by subtracting the integer $\min_{v \in V(G)} f(v)$ from all the labels of the vertices of the graph. Determining the radio number of a graph is an interesting yet difficult combinatorial problem with potential applications to FAP. So far it has been explored for a few basic families of graphs. The radio number of any hypercube was determined in [14] by using generalized binary Gray codes. Ortiz et al.[21] have studied the radio number of generalized prism graphs and have computed the exact value of radio number for some specific types of generalized prism graphs. For two positive integers $m \geq 3$ and $n \geq 3$, the Toroidal grids $T_{m,n}$ are the cartesian product of cycles $C_m$ with the cycles $C_n$. Morris et al. [20] have determined the radio number of $T_{m,n}$ and Saha et al. [23] have given exact value for radio number of $T_{m,n}$ when $mn \equiv 0\ (mod 2)$. The radio numbers of the square of paths and cycles were studied in [17, 18]. The first result on the radio number of trees was given by Chartrand et al. in [4, 5]. They studied the radio labelling problem for paths and cycles. Later, Liu and Zhu gave the exact radio number of paths in [16]. The lower bound for the radio number of trees and different necessary and sufficient condition to achieve the lower bound is given by Liu [15] and Bantva et al. [1]. For paths $P_n$, complete $m$ −ary trees, the exact values of radio number were determined in [16, 19]. The results for paths were generalized [16] to spiders, leading to the exact value of the radio number in certain special cases. In [22], Reddy et al. give an upper bound for the radio number of some special type of trees. In spite of these efforts, the problem of determining the exact value of the radio number for trees is still open, and it seems unlikely that a universal formula exists for all trees.

Inspired by the work in [15], in this article we give a lower bound for radio number of any trees in terms of number of vertices, diameter and the weight of maximum Hamiltonian path. As an improvement of this bound a new bound has been given for particular class of trees namely, trees of even diameter having two-degree centroid. It is also shown that there are many trees whose radio numbers coincides with these bounds. All the graphs considered in this paper are simple and connected, unless otherwise stated. A complete graph and a path on $n$ vertices are denoted by $K_n$ and $P_n$, respectively. A Hamiltonian path in a graph $G$ is a path through all of the vertices of the graph, visiting each vertex once and only once. A non-empty binary tree $T$ with $T_L$ and $T_R$ as its left and right subtree is called AVL tree if and only if $(a)$ $|h_L - h_R| \le 1$, where $h_L$ and $h_R$ are the heights of $T_L$ and $T_R$, respectively, and $(b)$ $T_L$ and $T_R$ are AVL trees. Complete binary trees are examples of AVL trees. In this article, we give a lower bound for radio numbers of arbitrary trees and shows its sharpness for some AVL trees.

**2. Preliminaries:**

**Definition 2.1:**

The *metric closure* of a connected graph $G = ((V(G), E(G))$, denoted by $G^c$, is the complete weighted graph on $V(G)$ in which weight of an edge $\{u, v\}$ is the distance between $u$ and $v$ in $G$. Note that the edge weights in $G^c$ satisfy the triangle inequality. The weight of a subgraph $H$ of $G^c$, denoted by $w(H)$, is the sum of weights of all edges in $H$.

**Definition 2.2:**

Let $T$ be any tree. The measure of *separability* of a vertex $v \in V(T)$, denoted by $\beta_T(V)$, is the size of maximum component of $T - \{v\}$. A vertex is called *centroid* if it has minimal separability over all vertices in $T$. Let $T$ be a tree with centroid $T$. Define the level of $u \in V(T)$ (with respect to $S$) by

$$L(u) = d(S, u).$$

A vertex $u$ of $T$ is in level $\ell$ if $L(u) = \ell$. For distinct $u, v \in V(T)$, define
$\phi(u, v) :=$ length of the common part of the paths of $T$ from $S$ to $u$ and $v$.

**Lemma 1:**

*Let $T$ be a tree rooted at $r$. Then for distinct $u, v \in V(T)$ the following $(a) - (b)$ hold.*
$(a)$ $d(u, v) = L(u) + L(v) - 2\phi(u, v)$
$(b)$ $\phi(u, v) = 0$ *if and only if $r \in \{u, v\}$ or $u$ and $v$ belongs to the different branches.*

**Lemma 2:**

*For an $n$ −vertex tree $T$, the following $(a) - (c)$ hold.*
$(a)$ *If a vertex $v$ is a centroid, then* $\beta_T(v) \le \left\lfloor \frac{n}{2} \right\rfloor$
$(b)$ *A tree with odd number of vertices has exactly one centroid.*
$(c)$ *A tree $T$ with even number of vertices has two centroids $S_1$ and $S_2$ which are neighbours and*

$$\sum_{u \in V(T)} d(S_1, u) = \sum_{u \in V(T)} d(S_2, u)$$

**Lemma 3:**

*Let $S$ be a centroid of an $n$-vertex tree $T$. Then there exist a sequence $u_0, u_1, \ldots, u_{n-1}$ of vertices of $T$ such that no two consecutive vertices are in same branch of $T - S$.*

**Notation 2.1:**

For a centroid $S$ of a tree $T$, we called $\gamma$ *an alternating sequence* of vertices in $T$ if no two consecutive vertices are in same branch of $T - S$. The set of all such $\gamma$ is denoted by $\Gamma$.

**Definition 2.3:**

A tree $T$ is said to be *tree having two degree centroid* or *two degree centroid tree* if it has at least one centroid of degree two.

**Definition 2.4:**

For a given tree , *a maximum weight Hamiltonian path* is a Hamiltonian path in $T^c$ of maximum weight. For $u, v \in V(T)$, the *$uv$-maximal Hamiltonian path* in $T^c$ is a maximum weight Hamiltonian path in $T^c$ whose end vertices are $u$ and $v$. Let us denote $w^*_{uv}(T^c)$ and $w^*(T^c)$ be the weight of $uv$ − maximal weight Hamiltonian path and maximum weight Hamiltonian path in $T^c$ respectively. Finding the weight of a maximum Hamiltonian path in $G^c$ is NP-hard. Lemma 4 gives the weight of a maximum Hamiltonian path in $T^c$ for any tree $T$. This lemma also describes the weight of a $uv$ −maximal weight Hamiltonian path in $T^c$.

**Lemma 4:**

*Let $T$ be an $n$ −vertex tree with centroid $T$. Then the following are hold:*

(a) $w_{uv}^*(T^c) = 2\sum_{u\epsilon V(T)} d(S,u) - d(u,S) - d(v,S)$

(b) $w^*(T^c) = 2\sum_{u\epsilon V(T)} d(S,u) - 1$

**Proof:**

$(a)$ For any Hamiltonian path $P: u = u_0u_1u_2 \dots u_{n-1} = v$ in $T^c$, the weight of the path is

$$w(P) = \sum_{i=0}^{n-2} d(u_i,u_{i+1}) \le \sum_{i=0}^{n-2}[d(S,u_i) + d(S,u_{i+1})] \quad (2)$$
$$= 2\sum_{i=0}^{n-2}[d(S,u_i) - d(S,u_0) - d(S,u_{n-1})] \quad (3)$$
$$= 2\sum_{i=0}^{n-2}[d(S,u_i) - d(S,u) - d(S,v)] \quad (4)$$

Equality Occurs in (2) if $u_i$ and $u_{i+1}$ are in different branches of $T - S$ and such types of choices of $u_i^{'}s$ are possible due to the Lemma 3. hence the part $(a)$ of this lemma is proved.

$(b)$ The maximum value of $w_{uv}^*(T^c)$ in $(b)$ will occur if we take $u = S$ and $v$ is a vertex adjacent to $S$. Thus the weight of a maximum weight Hamiltonian path in $T^c$ is equals to $2\sum_{u\epsilon V(T)} d(S,u) - 1$.

**3. Main Result:**

**3.1 Lower Bound for General Trees:**

The theorem below gives a lower bound for radio number of arbitrary trees.

**Theorem 3.1:**

*Let $f$ be a radio labeling of an $n-$vertex tree $T$ with first and last labeled vertices $u$ and $v$, respectively. Then*

$$\text{span}(f) \ge (n-1)(\text{diam}(T)+1) - 2w_{uv}^*(T^c)$$

*where $w_{uv}^*(T^c)$ is the weight of $uv$ $-$maximal weight Hamiltonian path.*

**Proof:**

Let $d$ be the diameter of $T$. Since $f$ is a radio labelling of $T$, $f$ induces a linear order $u = u_0, u_1, \dots, u_{n-1} = v$ of the vertices of $T$ such that $f(u) = f(u_0) < f(u_1) < f(u_2) < \cdots < f(u_{n-1}) = f(v)$. Then $uv$ $-$span of $f$ is given by

$$span(f_{uv}) = f(u) - f(v) = \sum_{i=0}^{n-2}[f(u_{i+1}) - f(u_i)]$$
$$\ge \sum_{i=0}^{n-2}[d + 1 - d(u_i,u_{i+1})]$$
$$\ge (n-1)(d+1) - \sum_{i=0}^{n-2} d(u_i, u_{i+1})$$
$$\ge (n-1)(d+1) - \sum_{i=0}^{n-2}[d(S,u_i) + d(S,u_{i+1})]$$
$$= (n-1)(d+1) - 2w(T) + d(S,u) + d(S,v)$$
$$= (n-1)(d+1) - 2w_{uv}^*(T^c)$$

Hence we obtain the result. The following corollary is due to Liu [15] that represents a lower bound for radio number of an arbitrary tree.

**Corollary 1:**

*Let $T$ be an $n$ $-$vertex tree with diameter $d$ and centroid $S$. Then*

$$rn(T) \ge (n-1)(d+1) - 2w(T) + 1.$$

**3.2. Radio Number of AVL Trees:**

A full $m-$ary tree $T_{m,h}(m \ge 2)$ is a rooted tree such that each vertex of degree greater than one has exactly $m$ children and all degree-one vertices are of equal distance (height) to the root. The lower bound given in Theorem 3.1 agrees with the exact value of radio number for full $m-$ary trees (which was settled by different approach in [19]). Let $T_h$ be a complete binary tree of height $h$. After deletion of some pendant vertices from $T_h$ the resulting graph again forms an AVL tree. Now we find an optimal radio labelling of some special type of AVL trees with the help of Theorem 3.1. First we give an arrangement of vertices of full binary binary tree $T_h$ of height $h$.

***Vertex-indices of $T_h$:***

We represents the vertices of $T_h$ by using binary numbers. First we denote root vertex of $T_h$ by $r$ and then two children of $r$ by $v_0$ and $v_1$. The children of $v_0$ by $v_{00}$ and $v_{01}$; and $v_1$ by $v_{10}$ and $v_{11}$. Inductively, two children of a vertex $v_{i_1,i_2,i_3,\dots,i_l}$ $(0 \le i_1, i_2, i_3, \dots, i_l \le 1$ *and* $l \le h$ level $l$ of full binary tree $T_h$ are indexed by $v_{i_1,i_2,i_3,\dots,i_l,i_{l+1}}$ where $i_{l+1} \epsilon \{0,1\}$. Now we introduce another index scheme for $T_h$. We first index root vertex by $u_0$ and then index other vertex bottom-up starting from level $h$. More precisely, for any vertex $v_{i_1,i_2,i_3,\dots,i_l}$ we rename $u_j = v_{i_1,i_2,i_3,\dots,i_l}$ where

$$j = 1 + i_1 + 2i_2 + \cdots + 2^{l-1}i_l + \sum_{l+1 \le t \le h} 2^t$$

Thus we have an arrangement of vertices of $T_h$ as $u_0, u_1, u_2, \dots, u_{n-1}$ where $n = 2^{h+1} - 1$ (number of vertices of $T_h$ ). It is also noted that the set $P = \{u_1, u_2, \dots, u_{2^h}\}$ is the set of all pendant vertices of $T_h$. Construct a symmetric partitions of $P$ with the sets $B_i = \{u_{i+4j} : 0 \le j \le 2^{h-2} - 1,$ where $i = 1,2,3,4$. After deletion $r$ number of vertices from each $B_i, 1 \le i \le 4$ resulting trees will form an AVL tree and we denote this tree by $A_h(r)$. Now we give an optimal radio labelling scheme for $A_h(r)$. For $1 \le i \le 4$, let $D_i$ denotes the collection of $r$ number of deleted vertices from $B_i$ and $S_i = \{u_{i+4j} \epsilon D_i : 0 \le j \le 2^{h-3} - 1\}$. If $u_k \epsilon S_i$, then the index $2^h + k$ will be replaced by $k$. Again if $u_k \epsilon D_i \setminus S_i$ and $u_{k-2^{h-1}} \notin S_i$, then the index $2h-1+k$ will be replaced by $k$. Finally, if $u_j \in V(A_h(r))$ with $j \geq 2h+1$, then the index $j$ will be replaced by $j-4r$. Define a radio labelling $f: V(A_h(r)) \to \{0,1,2,\dots\}$ as follows

$$f(u_0)=0$$
$$f(u_{j+1})=f(u_j)+2h+1-\{L(u_j)+L(u_{j+1})\}+\delta_j, \qquad 2\leq j\leq n-2$$

where $\delta_j=1$, if for three consecutive vertices $u_{j-1}, u_j, u_{j+1}$ the middle one is in level $h$ i.e. $L(u_j)=h$ and $\delta_j=0$ otherwise. In Figure 1, an optimal radio labelling of $A_4(4)$ has been presented.

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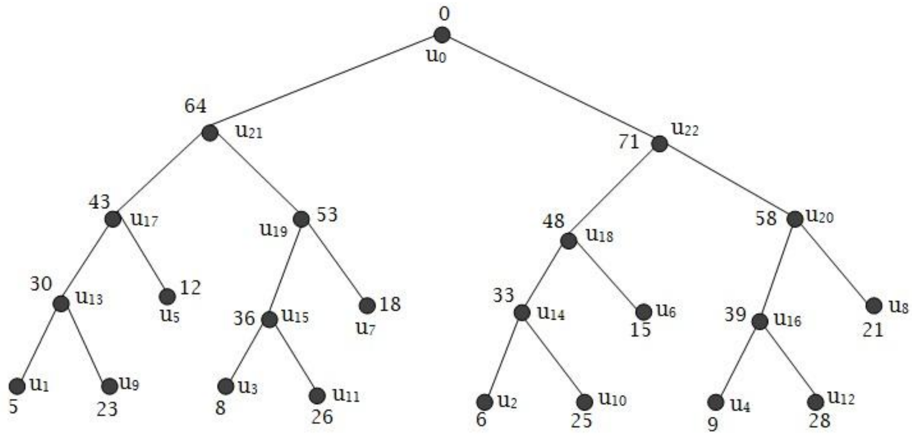


Figure 1: An optimal radio labelling of $A_4(4)$ with span $71$.

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