International Journal of Applied and Advanced Scientific Research (IJAASR)
Impact Factor: 5.655, ISSN (Online): 2456 - 3080
(www.dvpublication.com) Volume 5, Issue 1, 2020

ON A DIOPHANTINE EQUATION X^3-1/Y^3-1=Z^2

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Cite This Article: Balchandar Reddy Sangam, "On A Diophantine Equation x^3-1/y^3-1=z^2", International Journal of Applied and Advanced Scientific Research, Volume 5, Issue 1, Page Number 32-33, 2020.

Abstract:

In this paper, a Diophantine equation given by Gary Walsh in an advanced workshop (May 14-16, 2007), on the solvability of Diophantine equations, held at the Lorentz Center of Leiden University, The Netherlands, is studied. The problem goes as "Are there infinitely many positive integer solutions to $\frac{x^3-1}{y^3-1} = z^2$"

Key Words: Diophantine Equations, Multivariate polynomials

Introduction:

Diophantine equation is a polynomial equation which takes only integer values. They exist in two forms, that are linear and non-linear. In general, solving a non-linear diophantine equation is found to be difficult. There are various non-linear diophantine equations which are yet to be solved as of June 2020. The problem this article addresses is one of those unsolved problems on diophantine equations.

The 22 problems on Diophantine equations given in the workshop are considered as the most wanted Diophantine equations. In 2008, the solutions to a couple of equations, $x^2 - x = y^5 - y$ and $\binom{x}{2} = \binom{y}{5}$were given by Y. Bugeaud [5]. In 2009, Samir Siksek [4] has also provided a solution to an equation given in the workshop.

An Elementary Approach to the solution of a diophantine equation $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{2}$ was presented by Jozsef Sandor [1]. In 2017, P. Reka [3] has published an article on finding solutions to the equations $10^x + 2^y = z^2$ and $10^x + 3^y = z^2$.

Main Results:

It is obvious that the equation has infinitely many trivial solutions that are positive, but we will investigate whether there are any non-trivial positive integer solutions.

Theorem:

There are infinitely many positive integer solutions to the equation $\frac{x^3-1}{y^3-1} = z^2$, all of which are trivial.

Proof:

The equation can be written as $\frac{(x-1)(x^2+x+1)}{y^3-1} = z^2$

There are 5 possible ways in which the numerator and denominator of the expression $\frac{(x-1)(x^2+x+1)}{y^3-1}$ can be related. Solutions to each case are computed individually, as shown below.

Case 1: $x - 1 = y^3 - 1,\ x^2 + x + 1 = z^2.$

For any $x \in N$, $x^2 + x + 1$ cannot be represented as a perfect square, because $x^2 + x + 1$ falls in between the two consecutive perfect squares x^2 and $(x+1)^2$. Since $x^2 + x + 1$ cannot be represented as a perfect square, there is no value of z for which the equation holds true. Hence, the equation of this form do not have any positive integer solutions.

Case 2: $x^2 + x + 1 = y^3 - 1,\ x - 1 = z^2.$

Substituting $x = z^2 + 1$in the equation$x^2 + x + 1 = y^3 - 1$, we obtain $(z^2+1)^2 - (z^2+1) = y^3 - 1$. After simplification we have $z^4 + z^2 + 1 = y^3$.

The equation can be written as $(z^2 - z + 1)(z^2 + z + 1) = y^3$.

In the prime factorisation of y^3, the exponent of every prime is a multiple of 3. Since, $z^2 - z + 1$ and $z^2 + z + 1$ are co-primes, none of these primes occurs in both the factorisation $z^2 - z + 1$ and $z^2 + z + 1$. So both $z^2 - z + 1$ and $z^2 + z + 1$ are perfect cubes. So, $z^2 - z - 1 = y_1^3$ and $z^2 + z + 1 = y_2^3$ where $y_1^3 y_2^3 = y^3$.

According to [2], The only positive integer solutions to $z^2 + z + 1 = y_1^3$ are $(y_1, z) = (1,0), (7,18)$ and $z^2 - z + 1 = y_2^3$ are $(y_2, z) = (1,1), (7,19)$. As there is no z value for which both the expressions $z^2 - z + 1$ and $z^2 + z + 1$ are proper cubes. We can conclude that there are no solutions for this case.

Case 3: $k(x-1) = y^3 - 1,\ \frac{x^2+x+1}{k} = z^2, k \in N$

Substituting $x = \frac{y^3-1}{k} + 1$ in the equation $\frac{x^2+x+1}{k} = z^2, k \in N$

$$\frac{(y^3 - 1 + k)^2}{k^2} + \frac{y^3 - 1 + k}{k} + 1 = kz^2$$

$$y^6 + y^3(3k - 2) + (3k^2 - 3k + 1 - k^3 z^2) = 0$$

Considering the equation to be of the form $ax^2 + bx + c = 0$, to solve for y^3. For there to be an integer value of y which satisfies the equation, the determinant of the equation (D): $b^2 - 4ac$ should be a perfect square. Determinant to the equation is given as,

$$(3k-2)^2 - 4(3k^2 - 3k + 1 - k^3z^2)$$
$$4k^3z^2 - 3k^2$$
$$k^2(4kz^2 - 3)$$

As D cannot be represented as a perfect square, y does not have any integer solutions, which indicates that there are no positive integer solutions.

Case 4: $k(x^2 + x + 1) = y^3 - 1$**,** $\frac{x-1}{k} = z^2$**,** $k \in N$

Substituting $x - 1 = kz^2$ in the equation $k(x^2 + x + 1) = y^3 - 1$, we have

$$k(k^2z^4 - kz^2 + 1) = y^3 - 1$$
$$k^3z^4 - k^2z^2 + k + 1 = y^3$$

As $k^3z^4 - k^2z^2 + k + 1$ cannot be factored further, we conclude that there are no values of k, z and y for which the equation stands true. Hence, there are no positive integer solutions.

Case 5: $x^3 - 1 = y^3 - 1$**,** $z^2 = 1$**.**

This case gives us the trivial solution, that is$(n, n, 1); n \in N - \{1\}$**.**

Future Scope:

One can certainly look into other unsolved problems from the list of 22 open problems on diophantine equations given in the workshop. Another interesting problem that one could extend this work to is 'Find all integersolutions to $x^4 + x^2 + y^4 + y^2 = z^4 + z^2$'. The problems can be solved in a similar way by understanding the nature of perfect squares and prime factorisations of higher degree polynomials.

Conclusion:

Surprisingly the equation takes only the trivial solutions which are of the form $(n, n, 1); n \in N - \{1\}$. There are no non-trivial positive integer solutions to the given equation.

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