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ON DISTANCE MATRICES AND LAPLACIANS

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Abstract:

Distance matrices of graphs, particularly trees, have been investigated to a great extent in the literature. An early, remarkable result in this context concerns the determinant of the distance matrix of a tree: Graham and Pollack [3] showed that ifTis a tree on *n* vertices with distance matrix D, then the determinant of D is $(-1)^{n-1}(n-1)2^{n-2}$, and thus is a function of only the number of vertices; that paper also discusses the inertia ofD. (Recall that for symmetric matrix M, its inertiais the triple of integers($n_+(M)$, $n_0(M)$, $n_-(M)$) where $n_+(M)$, $n_0(M)$ and $n_-(M)$ denote the number of positive eigenvalues of *M*, the multiplicity of 0 as an eigenvalue of M, and the number of negative eigenvalues of M, respectively.)

Key Words: Matrics, Tree, Graphs, Symentric & Multipilicity

Introduction:

In subsequent work, Graham and Lovasz [4] obtained a formula for D^{-1} amongother results. In Section 2 we extend Graham's and Lovasz's formula for D^{-1} tothe case of a weighted tree. We also obtain an extension of the Graham and Pollack determinantal and inertial formulae to the weighted case. In Section 3 we further extend these results to distance matrices arising from unweighted uni cyclic graphs. Suppose that we have a weighted graph G= (V, E) with nvertices and medges, and that we assign an orientation to each edge of G. The associated (vertex - edge) incidence matrixQof Gis the n× mmatrix defined as follows. In Section 4 we investigate a perturbation problem for distance matrices arising from weighted trees. Let Dbe a distance matrix arising from a weighted tree and let L be a Laplacian matrix of any weighted graph G. For $\in > 0$, we consider perturbations ofD^{-1} of the form $\in D^{-1} - L$ and show that matrices of this form areinvertible and have a nonnegative inverse.

Basic Concepts:

Definition 1: A graph is a tuple G = (V, E) where v is a (finite) set of vertices and E is a finite collection ofedges. The set E contains elements from the union of the one and two elements subsets of V. That is, each edge is either a one or two element subset of V.

Definition 2: Let G = (V, E). A graph H = (V^1, E^1) is a subgraph of G if $V^1 \subseteq V$ and $E^1 \subseteq E$. The subgraph H is proper if$V^1 \subsetneq V$ or $E^1 \subsetneq E$.

Definition 3: A graph G = (V, E) is a simple graph. If G has no edges that are self - loops and if E is a subset of two element subset of V; (i.e) G is not a multi - graph.

Definition 4: An edge of a graph that joins a vertex to itself is called a loop. A loop is an edge $e = v_iv_i$.

Definition 5: If two vertices of a graph are joined by more than one edge that these edges are called multiple edges.

Definition 6: If G = (V, E) is a graph and v ∈ V and e = {v}, then edge e is called a self - loop. That is, any edge that is a single element subset of V is called a self - loop.

Definition 7: A graph G = (V, E) is a multigraph if there are two edges e_1 and e_2 in E so that e_1 and e_2 are equal assets. That is, there are two vertices v_1 and v_2 in V so that $e_1 = e_2 = \{v_1, v_2\}$.

Definition 8: A directed graph (diagraph) is a tuple G = (V, E) where V is a (finite) set of vertices and E is a collection of elements contained in V × V. That is, E is a collection of ordered pairs ofvertices.The edges in E are called directed edges to distinguish them from those edges in graph.

Definition 9: A weighted graph is a graph in which each edge is assigned a weight, which is a positive number.

Definition 10: An unweighted graph, or simply a graph, is thus a weighted graph with each ofthe edges bearing weight 1.

Distance Matrix of a Tree:

Theorem 1:

Let T be a weighted tree on n vertices with edge weights $\alpha_1, \ldots, \alpha_{n-1}$and let D be the corresponding distance matrix. Let L denote the Laplacian matrixfor the weighting of T that arises by replacing each edge weight by its reciprocal.For each i = 1,….n, let $d_i$be the degree of the vertex i, let $\delta_i = 2 - d_i$, and set$\delta^T = [\delta_1, \ldots, \delta_n]$. Then

$$D^{-1} = \frac{-1}{2} L + \frac{1}{2\sum_{i=1}^{n-1} \alpha_i} \delta\delta^T$$

Proof:

We use induction on n. For n = 2, we have $D = \begin{bmatrix} 0 & \alpha_1 \\ \alpha_1 & 0 \end{bmatrix}$, $L = \frac{1}{\alpha_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $\delta = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and the formula for D^{-1} follows readily. Now suppose we have a weighted tree on n vertices 1,2,…,n, and from a new weighted tree $\overline{T}$ on vertices 1,….n+1 by adding in a pendant vertex n+1, adjacent to vertex n with edge weight α_n. Let D, L and δ be the appropriate quantities for T and let $\overline{D}$, $\overline{L}$, and $\overline{\delta}$ be the corresponding quantities for $\overline{T}$. Letting e_n be the n^{th} standard unit basis vector in $\mathbb{R}^n$ and 1 be the all ones vector in $\mathbb{R}^n$,

$$\overline{L} = \left(\begin{array}{c|c} L + \frac{1}{\alpha_n} e_n e_n^T & \frac{-1}{\alpha_n} e_n \\ \hline \frac{-1}{\alpha_n} e_n^T & \frac{1}{\alpha_n} \end{array}\right), \quad \overline{\delta} = \left[\frac{\delta - e_n}{1}\right] \qquad \overline{D} = \left(\begin{array}{c|c} D & De_n + \alpha_n \\ \hline e_n^T D + \alpha_n 1^T & 0 \end{array}\right)$$

and,

Let $\sigma_{n-1} = \sum_{i=1}^{n-1} \alpha_i$ and $\sigma_n = \sum_{i=1}^{n} \alpha_i$, and note that $\frac{-1}{2}\overline{L} + \frac{1}{2\sum_{i=1}^{n}\alpha_i}\overline{\delta}\overline{\delta}^T = \frac{-1}{2}\overline{L} + \frac{1}{2\sigma_n}\overline{\delta}\overline{\delta}^T$

$$= \left(\begin{array}{c|c} \frac{-1}{2}\overline{L} - \frac{1}{2\alpha_n} e^n e_n^T + \frac{1}{2\sigma_n}(\delta\delta^T - \delta e_n^T - e_n\delta^T + e_n e_n^T) & \frac{1}{2\alpha_n} e^n + \frac{1}{2\sigma_n}(\delta - e_n) \\ \hline \frac{1}{2\alpha_n} e_n^T + \frac{1}{2\sigma_n}(\delta^T - e_n^T) & -\frac{1}{2\alpha_n} + \frac{1}{2\sigma_n} \end{array}\right)$$

$$= \left(\begin{array}{c|c} \frac{1}{2}L + \frac{1}{2\sigma_n}[\delta\delta^T - \delta e_n^T - e_n\delta^T] - \frac{\sigma_{n-1}}{2\alpha_n\sigma_n} e_n e_n^T & \frac{1}{2\sigma_n}\delta + \frac{\sigma_{n-1}}{2\alpha_n\sigma_n} e_n \\ \hline \frac{1}{2\sigma_n}\delta^T + \frac{\sigma_{n-1}}{2\alpha_n\sigma_n} e_n^T & -\frac{\sigma_{n-1}}{2\alpha_n\sigma_n} \end{array}\right)$$

From the induction hypothesis $D^{-1}1 = \frac{1}{2\sigma_{n-1}}\delta^T 1\delta = \frac{1}{\sigma_{n-1}}\delta$, So that $D\delta = \sigma_{n-1}1$.

Also from the induction hypothesis, $D^{-1} = \frac{-1}{2}L + \frac{1}{2\sigma_{n-1}}\delta\delta^T$.

We thus find that $-\frac{1}{2}\overline{L} + \frac{1}{2\sigma_n}\overline{\delta}\overline{\delta}\ T$

$$= \left(\begin{array}{c|c} D^{-1} - \frac{\sigma_n}{2\sigma_n\sigma_{n-1}}\delta\delta^T - \frac{1}{2\sigma_n}(\delta e_n^T + e_n\delta^T) - \frac{\sigma_{n-1}}{2\alpha_n\sigma_n} e_n e_n^T & \frac{1}{2\sigma_n}\delta + \frac{\sigma_{n-1}}{2\alpha_n\sigma_n} e^n \\ \hline \frac{1}{2\sigma_n}\delta^T + \frac{\sigma_{n-1}}{2\alpha_n\sigma_n} e_n^T & \frac{\sigma_{n-1}}{2\alpha_n\sigma_n} \end{array}\right)$$

Next, we note that

$$\overline{D} = \left(\begin{array}{c|c} D & De_n + \alpha_n 1 \\ \hline e_n^T D + \alpha_n 1^T & 0 \end{array}\right)$$

$$D^{-1} = \left(\begin{array}{c|c} D^{-1}\frac{\alpha_n}{2\sigma_n\sigma_{n-1}}\delta\delta^T - \frac{1}{2\sigma_n}(\delta e_n^T + e_n\delta^T) - \frac{\sigma_{n-1}}{2\alpha_n\sigma_n} e_n e_n^T & \frac{1}{2\sigma_n}\delta + \frac{\sigma_{n-1}}{2\alpha_n\sigma_n} e_n \\ \hline \frac{1}{2\sigma_n}\delta^T + \frac{\sigma_{n-1}}{2\alpha_n\sigma_n} e_n^T & -\frac{\sigma_{n-1}}{2\alpha_n\sigma_n} \end{array}\right)$$

Remark 1:

For an unweight tree on n $\geq$3vertices the (i, j) element of $D^{-1}$is zero if and only if i $\neq$j, one of the vertices i and j has degree 2,and i and j are not adjacent. To see this, observe that the (i, j) element of D^{-1} is zero if and only if $(2-d_i)(2-d_j)$ = (n-1) l_{ij} , so that our conditions are clearly sufficient.

Conversely, If i and j are adjacent, then l_{ij} = -1 so that (n - 1) l_{ij} = - (n - 1). But $(2-d_i)(2-d_j) \geq 1[2-(n-1)]$ = -(n-3) and the (i, j) elements ofD^{-1} is non-zero. Finally, If i = j, we see that science n $\geq$ 3, $(2-d)^2$ = (n-1)d hasno admissible positive integer solution for d.In particular, each row of D^{-1} corresponding to a vertex of degree 2 has 3 non zero entires a-1 on the diagonal and$\frac{1}{2}$ in each spot corresponding to an adjacent vertex.

In order to discuss the determinant and inertia properties of distance matricesof weighted trees (and later, of unicyclic graphs), we begin by considering the following some what larger class of weighted graphs. Let Gbe a weighted graph, andsuppose that we have a collection of weighted trees $B_1,\ldots,B_k$. We construct anew graph $\overline{G}$from Gand the trees $B_1,\ldots,B_k$ by adding, for each *i* = 1,…,k,a weighted edge between some vertex of $B_i$and some vertex of G. We say that the new graph $\overline{G}$ isconstructed by adding the weighted branches $B_1,\ldots,B_k$to G. Evidently both the weighted trees and the unicyclic graphs can be constructed in this fashion.

Theorem 2:

Let G be a connected weighted graph on n vertices with distance matrix D, and suppose that D**1** = d**1**. Form $\overline{G}$ from G by adding weighted branches to G on a total ofm new vertices, with positive weights $\alpha_1, \ldots, \alpha_m$ on the new edges. Let $\overline{D}$ be the distance matrix for $\overline{G}$. Then for each x $\in\mathbb{R}$, det($\overline{D}$+ xJ) $=(-2)^m$det(D) ($\prod_{i=1}^m \alpha_i)(1+\frac{nx}{d}+\frac{n}{2d}\sum_{i=1}^m \alpha_i)$. Further, $n_0(D)= n_0(\overline{D})$ and if, in addition, D is non singular, then $n_+(D) = n_+(\overline{D})$.

Proof:

We prove the statement regarding det($\overline{D}$+ xJ)by induction on m. For the case that m = 0, note that the eigen values of Dcan be written as $d=\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$, while the eigen values of D+ xJared+ nx and $\lambda_2, \ldots, \lambda_n$.It now follows that det(D+ xJ)= $d(1+\frac{nx}{d})\lambda_2,\ldots,\lambda_n$= det(D)$(1+\frac{nx}{d})$, as desired.

Suppose now the statement holds for some m$\geq$0 and that $\overline{G}$is formed as described by adding branches on m+ 1 new vertices to G. Then

$$\overline{D} + xJ = \left(\begin{array}{c|c} 1 & 0 \\ \hline e_{n+m}^T & 1 \end{array}\right)\left(\begin{array}{c|c} \widehat{D} + xJ & \alpha_{m+1}1 \\ \hline \alpha_{m+1}1^T & -2\alpha_{m+1} \end{array}\right)\left(\begin{array}{c|c} 1 & e_{n+m} \\ \hline 0 & 1 \end{array}\right)$$

Where $\widehat{D}$ is the distance matrix for the weighted graph on n + m vertices formed from $\overline{G}$ by deleting vertex n + m + 1. Thus,

det($\overline{D}$ + xJ) = $(-2\alpha_{m+1})$ det ($\widehat{D}$ + xJ + $\frac{\alpha_{m+1}}{2}$ J) = $(2\alpha_{m+1})(-2)^m$det(D)$(\prod_{i=1}^{m+1}\alpha_i)(1+\frac{n}{d}(x+\frac{\alpha_{m+1}}{2})+\frac{n}{2d}\sum_{i=1}^{m+1}\alpha_i)$,

The first equality following from schur's formula, and second from an application of the induction hypothesis. We now find readily thatdet($\overline{D}$ + xJ) = $(-2)^{m+1}$det(D)$(\prod_{i=1}^{m+1}\alpha_i)(1+\frac{nx}{d}+\frac{n}{2d}\sum_{i=1}^{m+1}\alpha_i)$

In order to deduce that $n_0(D)= n_0(\overline{D})$, we prove by induction on mthat for any x $\geq$0, $n_0(D)= n_0(\overline{D}+ xJ)$.

Note that the case m= 0 is straightforward, since the eigenvalues of $\overline{D}$+ xJconsist of the Perron value d+ nx, along with the remaining non - Perron eigen values of D. Next, suppose that the result holds for some m $\geq$ 0, and that $\overline{G}$is formed from G by adding branches on m+ 1 new vertices, with vertex n + m+ 1 pendant, and adjacent to vertex n+ mwith an edge of weight α_{m+1} between them.

Distance Matrix of a Unicyclic Graph:

Theorem 3:

Let D be the distance matrix for the cycle on2k+ 1 vertices. Then D^{-1} = $-2I - C^k - C^{k+1} + \frac{2k+1}{k(k+1)}J$.

Proof:

Since $e_1^T D$ = [0 , 1 , 2 ,…, k – 1, k , k , k – 1 ,…, 2 , 1], $e_{k+1}^T D$ = [k , k – 1 , k – 2 ,…, 1 , 0 , 1 , 2 ,…, k – 1, k],
$e_{k+2}^T D$ = [k , k , k – 1 ,…, 2 , 1 , 0 , 1 ,…, k – 2,k – 1],
We find that $(e_{k+1}^T + e_{k+2}^T + 2e_1^T)$ D = [2k , 2k + 1,…, 2k + 1].
It follows that $(C^k + C^{k+1} + 2I)$ D = (2k + 1) J – 1, and hence I = $((2k+1)J - I)^{-1}(C^k + C^{k+1} + 2I)$ D
= $(-I + \frac{2k+1}{4k(k+1)}J)(C^k + C^{k+1} + 2I)$ D = $(-2I - C^k - C^{k+1} + \frac{2k+1}{k(k+1)}J)$ D.

Corollary 1:

The distance matrix for a cycle on2k+ 1 vertices has just one positive eigenvalue.

Proof:

Evidently the eigenvalues for D^{-1} are $\frac{1}{k(k+1)}$ and - 2 – $(\cos\frac{2kj}{2k+1}\pi + \cos\frac{2(k+1)j}{2K+1}\pi)$,
j = 1, . . . ,2k, and only the first eigenvalue is positive.

Remark 2:

It is straightforward to see that $C^k + C^{k+1}$ is the adjacency matrix for a cycle of length 2k +1, where for each i =1, . . . ,2k+ 1,vertex i is adjacent to vertices i +k and i − k(taking those indices modulo 2k + 1).

It follows readily that the matrix $2I + C^k + C^{k+1}$ is permutationally similar to $2I + C + C^T$, so that det $(2I + C^k + C^{k+1})$ = det $(2I + C + C^T)$.A simple proof by induction on k shows that det $(2I + C + C^T) = 4$.Next we give formulae for the determinant and inertia for the distance matrix of an unweighted unicyclic graph having a cycle of odd length.

Theorem 4:

Let G be a unicyclic graph with2k+ 1 + m vertices and cycle length 2k+ 1. Let $\bar{D}$ be the distance matrix of G. Then det $(\bar{D}) = (-2)^m[k(k+1) + \frac{2k+1}{2} m]$, while the inertia of D is given by $(n_+(\bar{D}), n_0(\bar{D}), n_-(\bar{D}) = (1 , 0 , 2k + m)$.

Proof:

Let Dbe the distance matrix for the cycle on 2k+ 1 vertices, and observe that D1 = k(k+ 1)1. In particular, the hypothesis ofTheoremapplies to D, and $\bar{D}$is constructed from D as described in that theorem. From Theorem,

Let D be the distance matrix for the cycle on2k+ 1 vertices. Then $D^{-1} = -2I - C^k - C^{k+1} + \frac{2k+1}{k(k+1)} J$.We find that the eigenvalues of D consist of the Perron value k(k+ 1), along with the reciprocals of the eigenvalues of$-2I - C^k - C^{k+1}$ whose eigenvectors are orthogonal to 1. By Remark, $\det(-2I - C^K - C^{k+1}) = -4$ and clearly -4 is the eigenvalue of $-2I - C^k - C^{k+1}$ corresponding to 1, so that theremaining eigenvalues have product 1.

Hence det(D)= k(k+ 1). Applying Theorem, it now follows readily that Let G be a connected weighted graph on n vertices with distance matrix D, and suppose that D1 = d1.

Form $\bar{G}$ from G by adding weighted branches to G on a total of m new vertices, with positive weights $\alpha_1, \ldots, \alpha_m$ on the new edges. Let $\bar{D}$ be the distance matrix for $\bar{G}$. Then for each $x \in \mathbb{R}$, $\det(\bar{D} + xJ) = (-2)^m \det(D) \left(\prod_{i=1}^m \alpha_i\right) \left(1 + \frac{nx}{d} + \frac{n}{2d} \sum_{i=1}^m \alpha_i\right)$.

Further, $n_0(D) = n_0(\bar{D})$ and if, in addition, D is nonsingular, then $n_+(D) = n_+(\bar{D})$.

$$\det(D) = (-2)^m \left[k(k+1) + \frac{2k+1}{2} m\right]$$

From Corollary, The distance matrix for a cycle on2k+ 1 vertices has just one positive eigen value. We have $n_+(D) = 1$, and again applying Theorem,we find that$(n_+(\bar{D}), n_0(\bar{D}), n_-(\bar{D})) = (1, 0, 2k+ m)$.

Next we develop parallel results for the distance matrix of an unweighted unicyclic graph having a cycle of even length. We begin by analyzing the distance matrix for an unweighted cycle of even length.

Remark 3:

The distance matrix for the 2k - cycle is the circulant D = cric ([0 , 1 , 2 ,…, k ,k - 1 ,…, 2 , 1]).

In particular, for any $x \geq 0$, D + x J has perron value $k^2 + 2kx$ with perron vector 1.

Further, for each j = 1,…, 2k - 1, consider the 2kth root of unityχgenerates a non – Perroneigen value of D + xJ as follows:

$$\sum_{t=1}^k t\chi^t + \sum_{t=1}^k (k-t)\chi^{k+t} = k\sum_{t=1}^k \chi^{k+t} + (1-\chi^k)\sum_{t=1}^k t\chi^t = k\chi^{k+1}\frac{1-\chi^k}{1-\chi} + \frac{(1-\chi^k)\chi}{(1-\chi)^2}[-(k+1)\chi^k + k\chi^{k+1} + 1].$$

If j is even, then $\chi^k = 1$, so we get a 0 eigenvalue.

If j is odd, we get $\chi^k = -1$, so the eigen value becomes $2\left[-\frac{k\chi}{1-\chi} + \frac{\chi}{(1-\chi)^2}(k+2-k\chi)\right]$

$$= \frac{2\chi}{(1-\chi)^2}(-k + k\chi + k + 2 - k\chi) = \frac{4\chi}{(1-\chi)^2} = \frac{4}{|1-\chi|^4}\chi(1 - 2\bar{\chi} + \bar{\chi}^2) = \frac{4}{|1-\chi|^4}(\chi - 2 + \bar{\chi}) = -\frac{8}{|1-\chi|^4}\left[1 - \cos\frac{\pi j}{k}\right].$$

In particular, we see that for any $x \geq 0$, D + xJ has one positive eigenvalue, and nullity k – 1. The following result will be useful in discussing the inertia of the distance matrix for a unicyclic graph.

Lemma 1:

LetG_0be a connected graph with distance matrix D_0andsuppose that for all x> 0, $D_0 + xJ$ has a single positive eigen value (namely the Perron value). Form G_m from $G_0$by adding in unweighted branches at various vertices of G_0, on a total of m new vertices. If D_m is thecorresponding distance matrix,then $D_m + xJ$ has just one positive eigenvalue for any x> 0.

Proof:

We proceed by induction on m, and note that the case m= 0 is just the hypothesis on D_0. Note that for some i, we have that

$$D_{m+1} + xJ = \begin{pmatrix} D_m + xJ & D_m e_i + (x+1)\mathbf{1} \\ e_i^T D_m + (x+1)\mathbf{1}^T & x \end{pmatrix}$$

$$\text{Let } M = \begin{pmatrix} I & 0 \\ -e_i^T & 1 \end{pmatrix}$$

$$M(D_{m+1} + xJ)M^T = \begin{pmatrix} D_m + xJ & \mathbf{1} \\ \mathbf{1}^T & -2 \end{pmatrix} = A$$

So that, D_{m+1} + xJand A have the same inertia. Note also that A has a positive eigenvalueif and only if$D_m + (x + \frac{1}{2+\lambda})J$ has λ as a positive eigenvalue, which, by the induction hypothesis, must be the Perron value. Note that, necessarily, such a $\lambda$is a simple eigenvalue of Asince any Perron value is simple. Suppose that there are two positive eigenvalues λ_1 and λ_2 with $\lambda_1 > \lambda_2$ so that $\lambda_i$is the Perron value of$D_m + (x + \frac{1}{2+\lambda_i})J$, i = 1, 2.

Then, letting ρ denote the Perron value, we have $\lambda_1 = \rho(D_m + (x + \frac{1}{2+\lambda_1})J) < \rho(D_m + (x + \frac{1}{2+\lambda_2})J) = \lambda_2$.the inequality being strict since $\lambda_1 > \lambda_2$ which is a contradiction. Hence A can have only one positive eigenvalue so that D_{m+1} + xJ has just one positive eigenvalue.

Inverse Distance Matrix Perturbed By Alaplacian:

Lemma 2:

Let T be a weighted tree with n vertices, and suppose that each edge of T has been assigned an orientation. Let D be the distance matrix of T, and let L and Q denote the Laplacian matrix and incidence matrix, respectively, for the weighting of T that arises by replacing each edge weight by its reciprocal. Denote the degree of vertex i by d_i , let $\delta_i = 2 - d_i$, i = 1, . . . , n, and let$\delta^T = [\delta_1, \ldots, \delta_n]^T$.

Then thefollowing assertions hold:

1. $LD = \delta\mathbf{1}^T - 2I$, $DL = \mathbf{1}\delta^T - 2I$.
2. $Q^T DQ = -2I$.
3. $(D^{-1} - L)^{-1} = \frac{1}{3}D + \frac{1}{3}(\sum_{i=1}^{n-1} \alpha_i)J$.

Proof:

(i) As in Theorem, we denote the edge weights of T by $\alpha_1, \ldots, \alpha_{n-1}-1$. It follows from Theorem, Let T be a weighted tree on n vertices with edge weights. $\alpha_1, \ldots, \alpha_{n-1}$.and let D be the corresponding distance matrix. Let L denote the Laplacian matrix for the weighting of T that arises by replacing each edge weight by its reciprocal.For each i = 1, . . . , n, let d_i be the degree of the vertex i, let $\delta_i = 2 - d_i$, and set$\delta^T = [\delta_1, \ldots, \delta_n]$. Then $D^{-1} = -\frac{1}{2}L + \frac{1}{2\sum_{i=1}^{n-1}\alpha_i}\delta\delta^T$, that$-\frac{1}{2}LD + \frac{1}{2\sum_{i=1}^{n-1}\alpha_i}\delta\delta^T D = I$ and hence $LD = \frac{1}{2\sum_{i=1}^{n-1}\alpha_i}\delta\delta^T D - 2I$.

Also, as seen in the proof of Theorem, $D\delta = (\sum_{i=1}^{n-1}\alpha_i)\mathbf{1}$.and thus $LD = \delta^T - 2I$. The proof of the second part is similar.

(ii) By (i), $QQ^T D = \delta\mathbf{1}^T - 2I$ and, since Q has zero column sums, $QQ^T DQ = -2Q$. The result follows since Q has full column rank and thus admits a left - inverse. We remark that the assertion made in (ii) is well - known in the unweighted case.

(iii) The result follows immediately from (i) and the fact (see Theorem) that $D^{-1}1 = \frac{1}{\sum_{i=1}^{n-1}\alpha_i}\delta$.

Motivated by the above results, we conducted certain numerical experiments concerning $(D^{-1} - L)$, in which L was replaced by the Laplacian of an arbitrary connected graph, while D continued to be the distance matrix of a weighted tree. The results were interesting as well as unexpected and are presented in a sequence, culminating in Theorem, Let S be an n× n symmetric, positive semi definite matrix with $S\mathbf{1} = 0$ and suppose rank(S) = n–1.Let D be the distance matrix of a weighted tree on n vertices and let p ≥ 1 be an integer.

Then $(1 - tDS)^{-1}D > 0$, for all t >0.and Theorem, Let G be a weighted, connected, graph on n vertices and let $\tilde{L}$ be the Laplacian of G. Let D be the distance matrix of a weighted tree on n vertices.

Then (i) $(D^{-1} - \tilde{L})^{-1}$ is an entry wise positive matrix, and(ii) each entry of$F(\epsilon) = (\epsilon D^{-1} - \tilde{L})^{-1}$is decreasing in $\epsilon > 0$. Let Dbe the distance matrix of a weighted tree with at least two vertices. We will use the well - known fact that D, and any principal sub matrix of D of order at least 2, has exactly one positive eigenvalue, while the remaining eigen values are negative. A square matrix is said to be an N - matrixif all its principal minors are negative. A signature matrix is a diagonal matrix with ±1 on the diagonal.

Lemma 3:

Let S be an n× n symmetric, positive semi definite matrix with S1 = 0. Let D be the distance matrix of a weighted tree on n vertices. Then $D^{-1} - S$ is non singular and has1 positive and n− 1 negative eigen values.

Proof:

Using the notation and the conclusion of Theorem, $D^{-1} = \frac{1}{2} L + \frac{1}{2\sum_{i=1}^{n-1} \alpha_i} \delta\delta^T$

If $(D^{-1} - S)$ x= 0 for some vector x, then, using the preceding equation, $\delta^T \left(-\frac{1}{2} L + \frac{1}{2\sum_{i=1}^{n-1} \alpha_i} - \delta\delta^T - S\right) x = 0$ and thus $\delta^T\delta\ \delta^T x = 0$. Since $\delta^T\delta = 2n - 2(n-1) = 2$, then $\delta^T x = 0$. Thus $\left(-\frac{1}{2} L - S\right) x = 0$. Since L and S are positive semi definite, it follows that Lx= 0 and hence x must be a scalarmultiple of 1. Now since $\delta^T x = 0$ and $\delta^T\delta \neq 0$, we have that x = 0. Thus the matrix $D^{-1} - S$ is non singular. Similarly, for any $t \geq 0$, $(D^{-1} - t\,S)$is non singular and must have the same inertia for all $t \geq 0$. Since D^{-1} has 1 positive and $n - 1$ negativeeigenvalues, it follows that $D^{-1} - S$ has 1 positive and $n - 1$ negative eigenvalues.

Conclusion:

In this project on distance matrices and laplacians. Mainly concentrated about the some basic definitions have been discussed. Also we have discussed about some theorem. This dissertation discusses that distance matrix of a tree. We have obtained necessary and sufficient condition for the distance matrix of a unicyclic graph. Next this dissertation discusses that inverse distance matrix perturbed by a laplacian. The results discussed in the dissertation may be used to stay about various graph theory invariants.

References:

1. R. B. Bapat, The Laplacian matrix of a graph, Math. Student 65(1996) 214–223.
2. N. Dyn, W.A. Light, E.W. Cheney, Interpolation by piecewise -linear radial basis functions I, J.Approx. Theory 59 (1989) 202–223.
3. R. L. Graham, H.O. Pollack, On the addressing problem forloop switching, Bell System Tech. J. 50(1971) 2495–2519.
4. R. L. Graham, L. Lovász, Distance matrix polynomials of trees, Adv. Math. 29 (1) (1978) 60–88.
5. Edward J. Kaplan, Mathematical Programming and Games, John Wiley, 1982.
6. R. Merris, The distance spectrum of a tree, J. Graph Theory 14 (3) (1990) 365–369.
7. R. Merris, Laplacian matrices of graphs: a survey, Linear Algebra Appl. 197/198 (1994) 143–176.
8. T. Parthasarathy, G. Ravindran, N-matrices, Linear Algebra Appl. 139 (1990) 89–102.
9. L. Reid, X. Sun, Distance matrices and ridge function interpolation, Can. J. Math. 45 (6) (1993)1313–1323.
10. X. Sun, Solvability of multivariate interpolation by radial or related functions, J. Approx. Theory 72(3) (1993) 252–267.