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ON FUZZY IDEAL

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Abstract:

Fuzzy set theory was first propounded by Zadeh in 1965. Subsequently, Rosenfeld did pioneering work by applying fuzzy sets to the realm of group theory and formulated the concept of a fuzzy subgroup of a group with respect to the t-norm Minimum. Since then, a host of mathematicians have been engrossed in extending the concepts and results of abstract algebra to the broader framework of the fuzzy setting. So, motivated by the above, in this paper our aim is to study the properties of fuzzy ideal. Fuzzy ideal of a group were studied by Rosenfeld (1971) and Liu (1982) introduced the notions of fuzzy ideals of a ring. Here, we have derived some results over it following the notion of Liu.

Introduction:

In 1971 Rosenfeld defined the concept of fuzzy subgroup of a group. Since then several attempts have been made to investigate the properties of fuzzy subgroups. However, the process of fuzzification is relatively slow in ring theory. In this section our aim is to establish some interesting results for fuzzy ideals of ring and have also derived some results on the level subset of a ring R. Fuzzy subsets of set was introduced by Zadeh in 1965. Firstly Rosenfeld applied this concept to the theory of group oids and groups. Liu defined and studied the notion of fuzzy ideal of a ring whereas fuzzy ideal of a group were studied by Rosenfeld. Here, we will derive some results on fuzzy ideals of a ring.

Fuzzy Ideal:

A fuzzy subset μ of a ring R is called a fuzzy left (right) ideal of R if and only if

(i) $\mu(x-y) \geq \min\{\mu(x), \mu(x)\}$

and (ii) $\mu(x.y) \geq \mu(y)$ or $\mu(xy) \geq \mu(x)$) respectively for all $x, y \in$ R.

and the fuzzy subset μ of R be called a fuzzy ideal of R if for all x, y in R

(i) $\mu(x-y) \geq \min\{\mu(x), \mu(y)\}$

(ii) $\mu(xy) \geq \max\{\mu(x), \mu(y)\}$

In other words, if μ is a fuzzy left ideal as well as fuzzy right ideal then it will be called as fuzzy ideal.

Theorem: If μ be a fuzzy left (right) ideal of a ring R, then (i) $\mu(x) \leq \mu(0)$ (ii) $\mu(x) = \mu(-x)$ and (iii) $\mu(x+y) \geq \min(\mu(x), \mu(y))$ for al $x, y \in$ R.

Proof: Let (R, +, .) be a Ring and μ is a fuzzy left (right) ideal of R.

We have $\mu(0) = \mu(x-x) \geq \min\{\mu(x), \mu(x)\}\ \forall\, x \in R = \mu(x)$.

$\therefore\ \mu(0) \geq \mu(x)$

i.e $\mu(x) \leq \mu(0)$ - - - - (i)

Further, we have $\mu(0-x) \geq \min\{\mu(0), \mu(x)\} = \mu(x)$, using (i)

i.e $\mu(-x) \geq \mu(x)$ - - - - (ii)

On replacing x by –x, we get $\mu\{-(-x)\} \geq \mu(-x)$

i.e $\mu(x) \geq \mu(-x)$ - - - - (iii)

Hence from (ii) & (iii), we get

$\mu(x) = \mu(-x), \forall\, x \in R$ - - - - (iv)

We have, $\mu(x+y) = \mu\{x-(-y)\} \geq \min\{\mu(x), \mu(-y)\}$.

$= \min\{\mu(x), \mu(y)\}$, using (iv)

$\therefore\ \mu(x+y) \geq \min\{\mu(x), \mu(y)\}$

Theorem: If μ is a fuzzy ideal of a ring R, Then

(i) $\mu(0) \geq \mu(x) \geq \mu(1)$, for all $x \in R$

(ii) $\mu(x - y) = \mu(0)$ implies $\mu(x) = \mu(y)$.

Proof: Let μ be a fuzzy left (right) ideal of a ring R, Then for all x, y in R, we have

(i) $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$ and (ii) $\mu(xy) \geq \max\{\mu(x), \mu(y)\}$.

We have $\mu(0) = \mu(x - x) \geq \min\{\mu(x), \mu(-x)\}$

$= \min\{\mu(x), \mu(x)\}$, using Theorem 2.1 $= \mu(x)$

$\therefore \mu(0) \geq \mu(x)$

Further, we have $\mu(x) = \mu(x. 1) \geq \max\{\mu(x), \mu(1)\} = \mu(1)$

$\therefore \mu(x) \geq \mu(1)$.

Thus we get that $\mu(0) \geq \mu(x) \geq \mu(1)$.

Let x and y are any two distinct elements of R such that $\mu(x - y) = \mu(0)$. Then, we have

$\mu(x) = \mu(x - y + y) = \{\mu(x - y) + y\} \geq \min\{\mu(x - y), \mu(y)\}$,using (iii) of theorem 2.1

$= \min\{\mu(0), \mu(y)\} = \mu(y)$, using (i) of theorem 2.1

$\therefore \mu(x) \geq \mu(y)$ ---- (i)

We have, $\mu(y - x) = \mu\{-(x - y)\} = \mu(x - y)$ ---- (ii)

$\therefore \mu(y) = \mu(y - x + x) = \mu\{(y - x) + x\} \geq \min\{\mu(y - x), \mu(x)\}$

$= \min\{\mu(x - y), \mu(x)\}$, using (ii) $= \min\{\mu(0), \mu(x)\} = \mu(x)$

$\therefore \mu(y) \geq \mu(x)$ ---- (iii)

Thus, from (i) and (ii), we get $\mu(x) = \mu(y)$

Theorem: μ is a fuzzy ideal of a ring (R ,+ , .) if and only if

(i) $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$

(ii) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$

Proof: Let μ is a fuzzy ideal of a ring (R ,+ , .). Then, for all x, y in R, we have

(i) $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$ and (ii) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$

On replacing y by –y in (i), we get

$\mu\{x - (-y)\} \geq \min\{\mu(x), \mu(-y)\}$

i.e $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$, Since $\mu(-y) = \mu(y)$ in a fuzzy ideal

$\therefore \mu(x + y) \geq \min\{\mu(x), \mu(y)\}$

And $\mu(xy) \geq \max\{\mu(x), \mu(y)\}$ holds.

Conversely, we suppose that μ is a fuzzy subset of R and is such that

$\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$

$\mu(x y) \geq \max\{\mu(x), \mu(y)\}$

Since R is a ring hence $0 \in R$. Therefore, we have

$\mu(0 - x) = \min\{\mu(0), \mu(x)\} \Rightarrow \mu(-x) \geq \mu(x)$.

Again, we have

$\mu(0 + x) = \mu\{0 - (-x)\} \geq \min\{\mu(0), \mu(-x)\}$

$$\Rightarrow \mu(x) \geq \mu(-x).$$

$$\therefore \mu(-x) = \mu(x).$$

Finally, we have

$$\mu(x + y) = \mu\{x - (-y)\} \geq \min\{\mu(x), \mu(-y)\} = \min\{\mu(x), \mu(y)\}$$

$$\therefore \mu(x + y) \geq \min\{\mu(x), \mu(y)\}$$

Hence μ is a fuzzy ideal of R

Level Subset: Let μ be a fuzzy subset of a non-empty set X and let $t \in [0, 1]$. Then the set μ_t defined as $\mu_t = \{x \in X : \mu(x) \geq t\}$ is called a level subset of X with respect to μ .

Theorem: If μ is a fuzzy left (right) ideal of R and if $0 \leq t \leq \mu(0)$, then μ_t is a left (right) ideal of R.

Proof: Let R is a ring and μ is a fuzzy left ideal of R. we also assume that $0 \leq t \leq \mu(0)$. Since $\mu(0) \geq t$,hence from the definition of level subset, we get that $0 \in \mu_t$. Thus we conclude that $\mu \neq \varphi$.

Let x and y are any two elements of μ_t. Hence $\mu(x) \geq t$ and $\mu(y) \geq t$. Since μ is a fuzzy left ideal of R, therefore

$$\mu(x - y) \geq \min\{\mu(x), \mu(y)\} \geq \min\{t, t\} = t$$

$$\therefore \mu(x - y) \geq t \Longrightarrow x - y \in \mu_t.$$

Thus, we get that for all $x, y \in \mu_t \Longrightarrow x - y \in \mu_t$. Now we assume that $r \in R$. Since A is a fuzzy left ideal, hence $\mu(r x) \geq \mu(x)\} \geq t$

$$\text{i.e. } \mu(r x) \geq t \Longrightarrow r x \in \mu_t$$

Hence μ_t is a left ideal of R.

Following the same footsteps as above we can show that μ_t is also a fuzzy right ideal of R. Hence μ_t is a fuzzy ideal of R.

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