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ON TWO DIOPHANTINE EQUATIONS $10^x + 2^y = z^2$ and $10^x + 3^y = z^2$

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Abstract:

This paper concerns with the solution of the Diophantine Equations $10^x+2^y = z^2$ and $10^3+3^y = z^2$ in non-negative integers.

Key Words: Exponential Diophantine Equations & Catalan's Conjecture

1. Introduction:

By an exponential Diophantine equation (eDe) we mean an equation in equation in which the bases are integers, the exponents are unknown integers. In this article the exponents are nonnegative as well.

In 2002, J. Sandor ([5], [6]) studied two Diophantine equations $3^x+3^y = 6^z$ and $4^x+18^y = 22^z$. After that D, Acu (2007) [1] studied the Diophantine equation $2^x+5^y= z^2$. He found that this equation has exactly two solutions in non- negative integer $(x, y, z) \in \{(3,0,3),(2,1,3)\}$

In 2011, Suvarnamani [7] published a paper in finding non-negative solutions to the Diophantine equation of the form $2^x+p^y= z^2$ for every prime p,

2. Main Results:

In this study, we use Catalan's conjecture [4], It is proved that the only solution in integers a>1, b >1, x>1 and y>1 of the equation $a^x-b^y= 1$ is a=y=3 and b=x=2.

Theorem 2.1:

The Diophantine equation $10^x+2^y= z^2$ has unique solution in non-negative integers namely $(x, y, z)\in \{(0, 3, 3)\}$

Proof:

From the Diophantine equation $10^x+2^y= z^2$, we consider in 3 cases.

Case 1: x=0. We have $2^y = z^2-1 = (z-1)(z+1)$. Then there are two non –negative integers a and b such that $2^a= z-1$, $2^b = z+1$, a<b and a+b = y. so we have $2^a(2^{b-a}-1) = 2^b – 2^a = 2$. Therefore, $2^a = 2$ or a =1. It follows that z = 3 and $2^b = 4$. Then b = 2 and y =3. Therefore (0, 3, 3) is a solution of the Diophantine equation $10^x+2^y= z^2$.

Case 2: y = 0. We have $10^x = z^2-1 = (z-1)(z+1)$. Then there are two non-negative integers a and b such that $10^a = z-1$, $10^b = z+1$, a<b and a+b = x. Therefore, $10^a(10^{b-a}-1) = 10^b – 10^a = 2$. It follows that $10^a = 1$ or $10^a = 2$. $10^a = 2$ is impossible. Thus $10^a = 1$. That is a = 0. If a = 0 then z =2 and $10^b = 3$ which is impossible.

Case 3: x > 0 and y = 0. If we take x =2, then $2^y = z^2-10^2 = (z-10)(z+10)$. Then there are two non-negative integers a and b such that $2^a = z-10$, $2^b = z+10$, a<b and a+b = y. so we have $2^a (2^{b-a}-1) = 2^b – 2^a = 20$ Therefore $2^a = 1$ or $2^a = 2$ or $2^a = 4$.

If $2^a = 1$ then a = 0. It follows that z = 11. Then $2^b = 21$ which is impossible.
If $2^a = 2$ then a = 1. It follows that z = 12. Then $2^b = 22$ which is impossible.
If $2^a = 4$ then a = 2. It follows that z = 14. Then $2^b = 24$ which is also impossible.
Therefore the Diophantine equation $10^x+2^y= z^2$ has unique solution

$$(x,y,z) \in \{(0, 3, 3)\}$$

Theorem 2.2:

The Diophantine equation $10^x+3^y= z^2$ has unique solution in non-negative integers, namely $(x, y, z) \in \{(0, 1, 2)\}$

Proof:

From the Diophantine equation $10^x+3^y= z^2$, we consider in 3 cases.

Case 1: x = 0, We have $3^y = z^2-1 = (z-1)(z+1)$. Then there are two non-negative integers a and b such that $3^a = z-1$, $3^b = z+1$, a < b and a + b = y . So we have $3^a(3^{b-a} - 1) = 3^b – 3^a = 2$. Therefore, $3^a = 1$ or a = 0. It follows that z =2 and $3^b = 3$. That is b = 1 and y =1. (0, 1, 2) is a solution of the Diophantine equation $10^x+3^y= z^2$.

Case 2: y =0, we have $10^x= z^2-1 = (z-1)(z+1)$. Then there are two non –negative integers a and b such that $10^a = z-1$, $10^b = z+1$, a < b and a + b = x . Therefore, $10^a(10^{b-a} - 1) = 10^b – 10^a = 2$. It follows that $10^a = 1$ or $10^a = 2$. $10^a = 2$ is impossible. Thus $10^a = 1$. That is a = 0. If a = 0 then z = 2 and $10^b = 3$ which is impossible.

Case 3: x > 0 and y > 0. If we take x = 2. Then $3^y = z^2-10^2 = (z-10)(z+10)$. Then there are two. Non-negative integers a snd b such that $3^a = z-10$, $3^b = z+10$, a < b and a + b = y . So we have $3^a(3^{b-a} - 1) = 3^b – 3^a = 20$. Therefore, $3^a = 1$.

If $3^a = 1$ then a = 0. It follows that z = 11, then $3^b = 21$ which is impossible.
Therefore, the Diophantine equation $10^x+3^y= z^2$ has unique solution

$$(x, y, z) \in \{(0, 1, 2)\}$$

Conclusion:

In this paper, we have shown that Diophantine equations $10^x+2^y= z^2$ and $10^x+3^y= z^2$ have solutions in non-negative integer. To conclude one may search for other pattern of this Diophantine equation,

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