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PROPERTIES OF (i, j) SEMI-BITOPOLOGICAL GROUPS

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Abstract:

In this paper, we introduce and study a class of bitopologized groups called *(i, j)* - semi-bitopological groups.

1. Introduction:

If $(G, *)$ is a group, and τ_1 and τ_2 are topologies on G, then we say that $(G, *, \tau_1, \tau_2)$ is a bitopologized group. Given a bitopologized group G, a question arises about interactions and relations between algebraic and bitopological structures: which topological properties are satisfied by the multiplication mapping $m : G \times G \rightarrow G, (x, y) \rightarrow x * y$, and the inverse mapping $i : G \rightarrow G, x \rightarrow x^{-1}$. In this paper, we introduce and study a class of bitopologized groups called *(i, j)*-semi-bitopological groups.

2. Preliminaries:

Throughtout this paper $(G, *, \tau_1, \tau_2)$, or simply G, will denote a group $(G, *)$ endowed with the topologies τ_1 and τ_2 on G. The identity element of G is denoted by e, or e_G when it is necessary, the operation $* : G \times G \rightarrow G, (x, y) \rightarrow x * y$, is called the multiplication mapping and sometimes denoted by m, and the inverse mappling $i : G \rightarrow G, x \rightarrow x^{-1}$ is denoted by i. For a subset A of a topological space (X, τ_i), $i\ Cl\ (A)$ and $i\ Int\ (A)$ denote the closure of A and interior of A in (X, τ_i), respectively.

Definition 2.1: [2] *A subset S of a bitopological space (X, τ_1, τ_2) is said to be (i, j)-semiopen if $S \subset j\ Cl\ (i\ Int\ (S))$. The complement of an (i, j)-semiopen set is called an (i, j)–semiclosed set.*

Definition 2.2: [2] *The intersection of all (i, j)–semiclosed sets containing $S \subset X$ is called (i, j)–semiclosure of S and is denoted by (i ,j)-s Cl (S). The family of all (i, j)–semiopen (resp. (i ,j)–semiclosed) sets of (X, τ_1, τ_2) is denoted by (i, j)–SO (X) (resp. (i, j)-SC (X)). The family of all (i, j)–semiopen (resp. (i, j)–semiclosed) sets of (X, τ_1, τ_2) containing a point $x \in X$ is denoted by (i, j)–SO (X, x) (resp. (i, j)–SC (X ,x)).*

Definition 2.3: *A function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be :*

1. *(i, j)–semicontinuous [2] if $f^{-1}(V) \in$ (i, j)–SO (X) for every $V \in \sigma_i$.*
2. *(i ,j)-irresolute [1] if $f^{-1}(V) \in$ (i, j)-SO (X) for every $V \in$ (i, j)–SO (Y).*
3. *Pre-irresolute if $f(V) \in$ (i, j)-SO (Y) for every $V \in$ (i, j)–SO (X).*
4. *(i, j)–semihomeomorphism if f is bijective, (i, j)–irresolute and pre-(i, j)–semiopen.*

Lemma 2.4: *If $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is an (i, j)–semihomeomorphism then:*

1. *(i ,j)–s Cl (f (A)) = f ((i, j)-s Cl (A)) for all $A \subset X$;*
2. *(i j)–s Int (f (A)) = f ((i ,j)-s Int (A)) for all $A \subset X$.*

Definition 2.5: *A subset U of (X, τ_1, τ_2) is called an (i, j)–semineighbourhood of a point $x \in X$ if there exists an (i, j)–semiopen set V of (X, τ_1, τ_2) such that $x \in V \subset U$.*

3. Quasi Topological Groups:

Definition 3.1: *(G, o, τ_1, τ_2) is (i, j)–semi-bitopological group if (G, o) is a group, (G, τ_1, τ_2) is a bitopological space and left translation $L_x : G \rightarrow G$ for all $x \in G$ and right translation $R_x : G \rightarrow G$ for all $x \in G$ are (i, j)–semicontinuous and the mapping of inversion $i : G \rightarrow G$ defined by $i(x) = x^{-1}$ is (i ,j)–semicontinuous on G.*

Theorem 3.2:

Let (G, o, τ_1, τ_2) be an (i, j)–semi-bitopological group and β_e be the base at identity element e of G. Then:

1. *For every $U \in \beta_e$, there is an element $V \in$ (i, j)–SO (G, e) such that $V^{-1} \subset U$.*
2. *For every $U \in \beta_e$, there is an element $V\, o\, x \subset U$ and $x\, o\, V \subset U$ for each $x \in U$.*

Proof:

(1) Since (G, o, τ_1, τ_2) is an *(i, j)*–semi-bitopological group. There-fore, for every $U \in \beta_e$ there exists $V \in$ *(i, j)–SO (G, e)* such that $i(V) = v^{-1} \in U$ because the inverse mapping $i : G \rightarrow G$ is *(i, j)*–semicontinuous.

(2) Since (G, o, τ_1, τ_2) is and *(i, j)*–semi-bitopological group. Thus, for each $U \in \tau_i$ containing x, there exists $V \in$ *(i, j)–SO (G, e)* such that $R_x(V) = V^o x \subset U$.

Lemma 3.3:

Let A be a subset of an (i, j)–semi-bitopological group $(G, o, \tau, \mathcal{G})$. Then (i, j)–s Cl $(A^{-1}) \in$ i Cl (A^{-1}).

Proof:

Let $x \in ((i, j)\text{–}s\ Cl\ (A))^{-1}$ and let $U \in \tau_i$ containing x. Then, U^{-1} is an (i, j)–semiopen neighbourhood of x^{-1}. Since $x^{-1} \in (i, j)\text{–}s\ Cl\ (A)$, therefore, $U^{-1} \cap A \neq \emptyset$. This implies that $U \cap A^{-1} \neq \emptyset$. That is , $x \in i\ Cl\ (A^{-1})$ and so $((i, j)\text{–}s\ Cl\ (A))^{-1} \subset i\ Cl\ (A^{-1})$.

Remark 3.4: If (G, o, τ_1, τ_2) is a quasi bitopological group, then τ_i^{-1} is also a topology on G, called conjugate topology of τ_i. If (G, o, τ_1, τ_2) is a quasi bitopological group, then so is $(G, o, \tau_1^{-1}, \tau_2^{-1})$. Note that $\tau_i^{-1} = \{A \subset G: A^{-1} \in \tau_i\}$. We give the following important result:

Theorem 3.5:

Let (G, o, τ_1, τ_2) be an (i, j)–semi-bitopological group. If U is (i, j)–semiopen set in (G, o, τ_1, τ_2), then U^{-1} is (i, j)–semiopen in $(G, o, \tau_1^{-1}, \tau_2^{-1})$.

Proof:

The proof follows from the respective definitions.

Theorem 3.6:

If (G, o, τ_1, τ_2) is (i, j)–semi-bitopological group, then $(G, o, \tau_1^{-1}, \tau_2^{-1})$ is also an (i, j)–semi-bitopological group.

Proof:

Since (G, o) is a group, (G, τ_1, τ_2) is a bitopological space. Therefore, $(G, o, \tau_1^{-1}, \tau_2^{-1})$ is again a bitopological group. We need to prove that : $i = (G, o, \tau_1^{-1}, \tau_2^{-1}) \to (G, o, \tau_1^{-1}, \tau_2^{-1})$, and $L_x : (G, o, \tau_1^{-1}, \tau_2^{-1}) \to (G, o, \tau_1^{-1}, \tau_2^{-1})$ and $R_x : (G, \tau^{-1}) \to (G, \tau^{-1})$ are (i, j)–semicontinuous mappings. First, we show that L_x is (i, j)–semicontinuous. For this, let $V \in \tau_i^{-1}$. Then $V^{-1} = U \in \tau_i$. Since (G, o, τ_1, τ_2) is (i, j)–semi-bitopological group, the left (right) translation is (i, j)–semicontinuous. Hence $L_x^{\leftarrow}(U) \in (i, j)\text{–}SO\ (G, \tau_1, \tau_2)$, that is, $(U\ o\ x^{-1})^{-1} \in (i, j)\text{–}SO\ (G, \tau_1^{-1}, \tau_2^{-1})$, that is, $U\ o\ x^{-1} = V^{-1} o\ x^{-1} = (x\ o\ v)^{-1} = L_x^{\leftarrow}(V) \in (i, j)\text{-}SO\ (G, \tau_1^{-1}, \tau_2^{-1})$. This proves that L_x: $(G, o, \tau_1^{-1}, \tau_2^{-1}) \to (G, o, \tau_1^{-1}, \tau_2^{-1})$ is (i, j)–semicontinuous for every $x \in G$. Similarly, we can prove that right translation $R_x : (G, o, \tau_1^{-1}, \tau_2^{-1}) \to (G, o, \tau_1^{-1}, \tau_2^{-1})$ is (i, j)–semicontinuous. Trivially $i : (G, o, \tau_1^{-1}, \tau_2^{-1}) \to (G, o, \tau_1^{-1}, \tau_2^{-1})$ is continuous and hence (i, j)–semicontinuous. Hence $(G, o, \tau_1^{-1}, \tau_2^{-1})$ is also an (i, j)–semi-bitopological group.

Theorem 3.7:

If H is a discrete subgroup of an (i, j)–semi-bitopological group $(G, o, \tau_1^{-1}, \tau_2^{-1})$, then $(i, j)\text{–}s\ Cl\ (H)$ is a subgroup of G.

Proof:

Let $x, y \in (i, j)\text{–}s\ Cl\ (H)$. If U and V are respective open neighbourhoods of x and y, then $L_x^{-1}(U) = x^{-1}\ o\ U$ and $L_y^{-1}(U) = y^{-1}\ o\ U$ are (i, j)–semiopen neighbourhood of e. Since H is a discrete subgroup of an (i, j)–semi-bitopological group G, therefore, $x^{-1} o\ U \cap H \neq \emptyset$ and $y^{-1}\ o\ U \cap H \neq \emptyset$. Therefore, $(x\ o\ y^{-1}\ o\ x^{-1}\ o\ U \cap x\ o\ y^{-1}\ o\ H) \cup (x\ o\ y^{-1}\ o\ y^{-1}\ o\ V \cap x\ o\ y^{-1}\ o\ H) \neq \emptyset$. That is $W \cap x^{-1}\ o\ y^{-1}\ o\ H \neq \emptyset$, where $W = x\ o\ y^{-1}\ o\ x^{-1}\ U \cup x\ o\ y^{-1}\ o\ y^{-1}\ o\ V$ is an (i, j)–semiopen neighbourhood of $x\ o\ y^{-1}$. Thus, for each $x, y \in (i, j)\text{- } s\ Cl\ (H)$ implies that $x\ o\ y^{-1} \in (i, j)\text{–}s\ Cl\ (H)$. Hence $(i, j)\text{–}s\ Cl\ (H)$ is a subgroup of G.

Corollary 3.8:

If H is a discrete subgroup of an (i, j)–semibitopological group (G, o, τ_1, τ_2) then $i\ Cl\ (H)$ is a subgroup of G.

Theorem 3.9:

Let (G, o, τ_1, τ_2) be an (i, j)–semi-bitopological group. If A is open in G, then A o B and B o A are (i, j)–semiopen in (G, o, τ_1, τ_2) for any subset B of G.

Proof:

Let $x \in B$ and $Z \in A\ o\ x$ we show that z is (i, j)–semiinterior point of $A\ o\ x$, Let $z = y\ o\ x$ for some $y \in A = A\ o\ x\ o\ x^{-1}$. This implies that $y = z\ o\ x^{-1}$. Now $R_{x\text{-}1}: G \to G$ is (i, j)–semicontiunous, that is, for every open set containing $R_{x\text{-}1}(z) = z\ o\ x^{-1} = y$, there exists an (i, j)–semiopen set M_z containing z such that $R_{x\text{-}1}(M_z) \subset A$. This implies $M_z\ o\ x^{-1} \subset A$ or $M_z \subset A\ o\ x$. This implies z is (i, j)–semiinterior point of $A\ o\ x$. Thus $A\ o\ x$ is (i, j)–semiopen. This implies $A\ o\ B = \cup\ \bigcup_{x\epsilon B} A\ o\ x$ is (i, j)–semiopen in (G, o, τ_1, τ_2). Similarly we can prove that for every open set A of G and arbitrary subset B of G, $B\ o\ A$ is (i, j)–semiopen in an (i, j)–semi-bitopological group (G, o, τ_1, τ_2).

Definition 3.10: *A bijective mapping $f : (X, \tau_X, \mathcal{G}) \to (Y, \tau_Y, \mathcal{G})$ ia called quasi s- $\mathcal{G}$-homeomorphism if it is (i, j)–semicontinuous and (i, j)–semiopen.*

Theorem 3.11:

Let (G, o, τ_1, τ_2) be an (i, j)–semi-bitopological group. Then each left (right) translation L_x: $G \to G$ is $R_x : G \to G$ a quasi s- $\mathcal{G}$-homeomorphism.

Proof:

Since (G, o, τ_1, τ_2) is (i, j)–semi-bitopological group. Therefore, L_x: $G \to G$ is (i, j)–semicontinuous. So it is enough to show that L_x: $G \to G$ is (i, j)–semiopen. Let V be an open set in G. Then by Theroem 3.9, $L_g(V) = g\ o\ V \in SO\ (G)$. Hence L_x: $G \to G$ is an (i, j)–semiopen mapping.

Theorem 3.12:

Suppose that a subgroup H of an (i, j)–semi-bitopological group (G, o, τ_1, τ_2) contains a nonempty open subset of G. Then H is semiopen in G.

Proof:

By Theorem 3.11 for every $g \in H$, $R_g: G \rightarrow G$ is quasi s-homeomorphism. Let U be a nonempty open subset of G with $U \subset H$, then for every $g \in H$, the set $R_g (U) = U o g$ is (i, j)–semiopen in (G, o, τ_1, τ_2). Now $H = U \{ U o g : g \in H \}$ is (i, j)–semiopen in G being union of (i, j)–semiopen sets of G.

Definition 3.13: *A topological space (G, τ) is said to be quasi s-homeogeneous if for all $x, y \in G$, there is a quasi s- homemorphism f of the space G onto itself such that $f (x) = y$.*

Theorem 3.14:

If (G, o, τ_1, τ_2) is an (i, j)–semi-bitopological group, then every open subgroup of G is also (i, j)–semiclosed.

Proof:

Since (G, o, τ_1, τ_2) is an (i, j)– semi-bitopological group and H is an open subgroup of G. Then any left or right translation $x o H$ or $H o x$ is (i, j)– semiopen for each $x \in G$. So $Y= \{x o H: x \in G\}$ of all left cosets of H in G forms a partition of G, Thus Y is an (i, j)–semiopen covering of G. by disjoint (i, j)–semiopen sets of G. This gives $G\backslash H$ is union of (i, j)–semiopen sets and hence (i, j)–semiopen. This proves that H is (i, j)–semiclosed.

Corollary 3.15:

Every (i , j)–semi-bitopological group is a quasi s- homogeneous space.

Proof:

Let us take element x and y in (G, o, τ_1, τ_2), and put $z = x^{-1} o y$. Since $R_x: G \rightarrow G$ is a quasi s-homeomorphism of (G, o, τ_1, τ_2) and $R_z (x) = x o z = x o (x^{-1} o y) = e o y = y$. Hence (G, o, τ_1, τ_2) is quasi s-homogeneous space.

Lemma 3.16: *If $f: (X, \tau_X) \rightarrow (Y, \tau_Y)$ is (i, j)-semicontinuous and H is open subset of X, then $f_H : (H, \tau_H) \rightarrow (Y, \tau_Y)$ is (i, j)–semicontinuous.*

Theorem 3.17:

Every open subgroup H of an (i , j)–semi-bitopological group (G, o, τ_1, τ_2) is also an (i , j)–semi-bitopological group (called (i , j)–semi-bitopological subgroup) of G.

Proof:

Let (G, o, τ_1, τ_2) be an (i, j)–semi-bitopological group and H be an open subgroup of G. We need to prove that (H, o, τ_H) is an (i , j)–semi-bitopological group. For this, We show that $i : H \rightarrow H$, $L_x : H \rightarrow H$ and $R_x : H \rightarrow H$ are (i , j)–semicontinuous with respect to the relative topology. Since H is an open subgroup of G. Then by Lemma 3.16, $i_H : (H, \tau_H) \rightarrow (Y, \tau_Y)$, $L_H : (H, \tau_H) \rightarrow (Y, \tau_Y)$ and $R_H :(H, \tau_H) \rightarrow (Y, \tau_Y)$ are (i , j)–semicontinuous. This proves that (H, o, τ_H) is an (i, j)–semi-bitopological group.

Theorem 3.18:

Let $f : (G, o, \tau_G) \rightarrow (H, o, \tau_H)$ be a homomorphism of (i , j)–semi-bitopological groups. If f is (i , j)–irresolute at the natural (identity) element e_G, then f is (i , j)–semicontinuous on G.

Proof:

Let $x \in G$ be an arbitrary element. Suppose that W is an open neighbourhood of $y = f (x) \in H$. Since the left translation in H is an (i , j)–semicontinuous mapping, there is an (i , j)–semiopen neighbourhood V of the nautral element e_H of H such that $L_y (V)= y o V \subset W$. Since f is (i, j)–irresolute at e_G, therefore, $f (U) \subset V$ for some (i, j) –semiopen neighbourhood U of e_G in G. Since $f (U) \subset V$, now $y o f (U) \subset y o V \subset W$. This implies $(x o U) \subset W$. Since (G, o, τ_G) is an $(i \ j)$–semi-bitopoogical group, $x o U$ is (i , j)–semiopen in G. This proves that f is (i, j)–semicontinuous at x. Since x was the arbitrary element of G, f is (i ,j)–semiconitinuous on G.

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