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VIZING'S WEAKER CONJECTURE $(\delta, \Delta) = (7,15)$

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Abstract:

Vizing conjectured that G is a simple and Δ-critical graph with m edges then $2m \geq \Delta^2$.—In this paper–we prove the conjecture for graphs with $\delta = 7$ and $\Delta = 15$.

Key Words: Critical Graphs & Degree Sequence

1. Introduction:

Throughout this paper, $G = (V, E)$ is a graph with n vertices, m edges, maximum degree $\Delta(G)$, and a minimum degree $\delta(G)$. If $v \in V$, then $d_G(v)$ denotes the degree of a vertex v in G. Let n_j be the number of vertices of degree j in G. we use $\pi(G)$ to denote the valency list of G. Note that $n_j = 0$, then the factor j^{nj} customary omitted $\pi(G)$. If S, T denotes the set of major and minor vertices in G respectively, and $|[S, T]|$ denotes the set of edges in G with one end in S and the other end in T. Also $|[T]|$ denotes the number of edges in T. s(G) denotes the sum of degrees of minor vertices in G. Let c(G) denotes the closure of G, then C denotes the Hamilton cycle of G. A well known theorem of Vizing [8] states that: if G is a simple graph with maximum degree Δ, then the edge chromatic number $X'(G)$ of G is Δ or $\Delta+1$.A graph G is said to be of class1 if $X'(G) = \Delta$ and it is said to be of class2 if $X'(G) = \Delta+1$.G is said to be(edge chromatic)critical if it is connected, class2 and $X'(G-e) < X'(G)$ for every edge e. A critical graph G with maximum degree Δ is called Δ-critical.

Conjecture [7]: If G is a Δ-critical graph with n vertices m edges and maximum degree Δ then $m > \frac{1}{2}(n(\Delta-1)+3)$. Recognizing that conjecture is probably difficult to settle, Vizing remarks that he is enable to settle the simple problem.

Is it true if G is simple and Δ-critical then $m \geq \frac{\Delta^2}{2}$?

We refer this problem as the Vizing's weaker conjecture.

K. Kayathri[3] proved this conjecture for graphs with $2 \leq \delta \leq 5$.

M. Santhi [6] proved this conjecture for graphs with $\delta = 6$.

In the following results we study the structure of 15- critical graphs with $\delta = 7$ and $2m < \Delta^2$.

2. Known Results:

To prove our result, we require the following preliminary results and their consequences.

R1 [7]:Vizing's Adjacency Lemma (VAL). In a Δ- critical graph G if vw is an edge and $d(v) = k$,then w is adjacent with atleast $\Delta - k + 1$ other vertices of degree Δ.

R2 [2]:A graph G with order 2s+1 and maximum degree 2s-1 is in class 2 iff it has size at least $2s^2$-s+1.

R3 [1]: A graph G with order 2s+2 and maximum degree 2s-1 is in class 2 iff q(G)- $\delta(G) \geq 2s^2 - s + 1$, where q(G) denotes the size of G.

R4 [4]: If G has order 2s and maximum degree 2s-1,then G is in class 2.If G has order 2s+1 and maximum degre2s,then G is in class 2 iff the size of G is at least $2s^2$+1.

R5 [5]: There are no critical graphs with order 2s+2 and maximum degree 2s.

R6 [6]: Let G be Δ-critical graphs with $n = \Delta+1$ or $\Delta+2$. Then $2m \geq \Delta^2$.

R7 [6]: Let G be Δ-critical graphs with $n = \Delta+3$ and Δ odd. Then $2m \geq \Delta^2$.

R8 [6]: Let G be graph with $n_\Delta \leq \Delta+1$. If $s(G) = n_\Delta(\Delta - n_\Delta + 1) + 2k$, then $|[T]| \leq k$ and $d(v) \leq n_\Delta + k$ for all $v \in T$.

R9 [6]: Let G be graph with $n_\Delta \leq \Delta+1$.If $s(G) = n_\Delta(\Delta - n_\Delta + 1)$ then,

i) $|[T]| = 0$

ii) $|[S, T]| = n_\Delta(\Delta - n_\Delta + 1)$ and

iii) Every vertex in S has exactly $\Delta - n_\Delta + 1$ neighbours in T.

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R10 [6]: Let G be Δ-critical graphs with $n_\delta \geq n-\delta+2$ and $s(G)=n_\Delta(\Delta-n_\Delta+1)+2k$. Then [T]=k and $d(v)\leq n_\Delta+k$ for all $v\in T$.

R11 [6]: Let G be Δ-critical graphs with $n_\Delta=\Delta-4$ and $2m<\Delta^2$, then

i) $5\Delta-20\leq s(G)<4\Delta$.

ii) $0\leq|[T]|\leq 2$ and

iii) s(G) and Δ are of same parity.

R12 [3]: Let G be Δ-critical graphs with $n_\Delta=(\Delta-\delta+2)+l$, where $1\geq 0$. If $\Delta\geq(\delta-1-l)(\delta-2-l)$, then $2m\geq\Delta^2$.

If G is a Δ-critical graphs with $\delta=7$, then by VAL $n_\Delta\geq(\Delta-\delta+2)=\Delta-5$.

3. Theorems:

Lemma1:

Let G be a 15-critical graph with $\delta=7$ and $n_\Delta=\Delta-3$. Then $2m\geq\Delta^2$.

Proof:

By R6 and R7 it is enough to verify the result when $n_\Delta=\Delta+4$. By VAL, $n_\Delta=\Delta-\delta+2$.

Let $n_\Delta=\Delta-\delta+2+l$ where $l\geq 0$. Now $n_\Delta=\Delta-3$ and $\delta=7$ implies that $l\geq 2$.

When $l\geq 3$, $(\delta-1-l)(\delta-1-l)=(6-l)(5-l)$

$$\leq 3\times 2=6<\Delta$$

and hence by R12, $2m\geq\Delta^2$. When $l=2$, $(\delta-1-l)(\delta-1-l)=4\times 2=8<\Delta$ and by R7, $2m\geq\Delta^2$ if $\Delta\geq 12$.

Since $n_\Delta=\Delta-3, n_\Delta\geq\Delta+3$ and $2m\geq\Delta^2$, $\mathbf{n_\Delta+\delta(n-n_\Delta)}$, we have

$$\begin{aligned}2m &\geq(\Delta-\delta)n_\Delta+\delta_n\\ &\geq(\Delta-7)(\Delta+3)+7(\Delta+4)\\ &\geq\Delta^2-3\Delta+49\\ &\geq\Delta^2 \text{ if } \Delta\leq 16\end{aligned}$$

Hence the result.

Lemma 2:

Let G be a Δ- cricital graph with $\delta=7$, $n_\Delta\geq\Delta+r, r\geq 3$.Then $2m\geq\Delta^2$ if $4\Delta\leq 28+7r$.

Proof:

$$\begin{aligned}2m &\geq\Delta n_\Delta+7(n-n_\Delta)\\ &\geq\Delta(\Delta-4)+7(\Delta+r-\Delta+4)\\ &\geq\Delta^2-4\Delta+28+7r\end{aligned}$$

Thus, $2m\geq\Delta^2$ if $4\Delta\leq 28+7r$.

Lemma 3:

let G be a15-critical graph with $\delta=7$, $\Delta=15$ and $2m\geq\Delta^2$.Then

i) $n_\Delta=\Delta-4$ and Δ-5.

ii) $n_\Delta=\Delta+r, r=3,4$

iii) $0\leq|[T]|\leq 2$

iv) s(G) =55 or 57 or 59.

Proof:

i) By VAL $n_\Delta\geq\Delta-\delta+2\geq\Delta-5$. Also by lemma 1, when $n_\Delta=\Delta-4$ and Δ-5, $n_\Delta<\Delta-3$. Hence $n_\Delta=\Delta-4$ and Δ-5

ii) Let $n=\Delta+r, r=3,4$.By lemma 2, $2m\geq\Delta^2$ if $4\Delta\leq 28+7r$. Hence for 15, if, $r\geq 5$ we have $2m\geq\Delta^2$. Also by R6 and R7 $2m\geq\Delta^2$ if $n=\Delta+1$ and $\Delta+2$.

iii) By R11, $0\leq|[T]|\leq\frac{20-5}{2}$ and so $0\leq|[T]|\leq 2$.

iv) By R11, s(G) is odd . Also $6\Delta-20\leq s(G)<4s$ and so $55\leq s(G)<60$.

Lemma4:

If G is a Δ- critical graph with $\Delta = 15$, $\delta = 7$, $n = \Delta + 4$ and $n_\Delta = \Delta - 4$ then $2m \geq \Delta^2$.

Proof:

If possible let G be a Δ-critical graph with $\Delta = 15$, $\delta = 7$, $n = \Delta + 4$, $n_\Delta = \Delta - 4$ and $2m \geq \Delta^2$.

Then by R11, $55 \leq s(G) < 60$ and s(G) is odd and so s(G) is 55,57 or 59.

$$\text{Now} \quad s(G) \geq \delta n_\delta + (\delta + 1)(n - n_\Delta - n_\delta)\Delta$$
$$\geq (\delta + 1)(n - n_\Delta) - n_\delta$$

For $1 \leq n_7 \leq 4$,

$s(G) \geq (7+1)(19-11) - 4 \geq 60$, a contradiction.

Now for $5 \leq n_7 \leq 8$, the possible degree sequences of G are as follows :

i) $7^5\, 8^3\, 15^{11}$
ii) $7^6\, 8\, 9\, 15^{11}$
iii) $7^7\, 8\, 15^{11}$
iv) $7^7\, 10\, 15^{11}$

In all the cases , we get a contradiction in the following Lemmas (Lemma 5 and Lemm 6).

Hence the Lemma.

Lemma 5:

If G is a Δ-critical graph with $\Delta = 15, \delta = 7$, $n = \Delta + 4$, $n_\Delta = \Delta + 4, s(G) = 57$, then $2m \geq \Delta^2$.

Proof:

Assume the contrary that $2m < \Delta^2$.Then the only possible degree sequence is $\pi(G) = 7^7 8 15^{11}$, given in Lemma 4. But by $|[T]| \leq 1$. But by VAL, $|[T]| = 0$.

Now, $|[S,T]| = 57$ if $|[T]| = 0$.

Then in G, one of the following two cases arises:

i) $|[T]| = 0$, two major vertices have 6 minor neighbours and 9 major vertices have 5 minor neighbours.

ii) $|[T]| = 0$, one major vertex has 7 minor neighbours and 10 major vertices have 5 minor neighbours.

Now $\pi(G) = 7^7 8 15^{11}$.

Let v_1 be a vertex of degree 7.

Let D be a subset of $T/\{v_1\}$ with $|D| = 7$.

Let $D' = T/D$. Then $|D'| = 1$ and $v_1 \in D'$.

Let $D = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$ and $D' = \{v_1\}$

We shall fix the degrees of u_i's and v_i's accordingly.

Let $G_1 = G/D$. Now $|v(G_1)| = 13$. Then $\Delta(G_1) \leq 12$.

The number of vertices of degree is in G_1. Since we have deleted 6 vertices from G and $\Delta(G) = 15$, the major vertices in G are of degree ≥ 9 in G_1.

Hence $n'_8 + n'_9 + n'_{10} + n'_{11} + n'_{12} \geq n_\Delta(G) = 11$

$|V_1| = n'_8 + n'_9 + n'_{10} + n'_{11} + n'_{12} \geq n_\Delta(G) = 11$

Moreover $v_1 \in G_1$and $d_{G_1}(v_1) = 7$.

Let $v_1 = \{v \in V(G_1) : d_{G_1}(v) \geq 8\}$ and

$v_2 = \{v \in V(G_1) : d_{G_1}(v) < 8\}$.

Then $|V_1| = n'_8 + n'_9 + n'_{10} + n'_{11} + n'_{12} \geq n_\Delta(G) = 11$.

Let $v \in V_1$. Now for all $w \in V(G_1)$

$d_{G_1}(v) + d_{G_1}(w) \geq 8 + 6 = 14 \geq |V(G_1)|$.

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So in the closure of $c(G_1)$, every $v \in V_1$ is adjacent with every other vertex in G_1. Moreover for all $u \in V_2$, $d_{C(G_1)}(u) \geq |V_1| \geq n_\Delta(G) = 1$.

So, for every pair of vertices u and w in V_2, $d_{C(G_1)}(u) + d_{C(G_1)}(w) \geq 11 + 11 = 22 > |V(G_1)|$

So, $c(G_1)$ is complete and hence G_1 is Hamiltonian.

Let C be a Hamiltonian cycle of G_1.

Let $G' = G/E(C)$.

Since G is of class 2, G' is also of class 2.

$$\left.\begin{aligned} \text{Also} \quad & d_{G'}(u) = d_G(u) \text{ for } u \in D \text{ and} \\ & d_{G'}(v) = d_G(v) - 2 \end{aligned}\right\} \qquad (1)$$

In particular $d_{G'}(v_1) = 5$ and so $\delta(G_1) = 5$. Also $\Delta(G') = \Delta(G) - 2 = 13$

Let H be a 13-critical subgragh of G'.

Let h_i'' denote the number of vertices of degree is in H.

Let S', T' respectively denote the set of major and minor vertices in H.

We have $|S'| \leq n_\Delta(G) = 11$. By VAL, $n_\Delta \geq \Delta - S + 2$

We have
$$\begin{aligned} \delta(H) &\geq \Delta(H) - |S'| + 2 \\ &\geq 13 - 11 + 2 \\ &\geq 4 \end{aligned}$$

$\delta(H) = 5$

Now $|S'| = 11$. Note that $d_H(v_i) = 5$ and so v_1 has 5 major neighbourhood in cases (i) and (ii)

$$\left.\begin{aligned} \text{Then in (i), } & |[S', T']| \leq (5\times 4) + (4 \times 5) + (2 \times 6) = 52 \\ \text{in (ii), } & |[S', T']| \leq (5\times 4) + (5 \times 5) + (1 \times 7) = 52 \end{aligned}\right\} \qquad (2)$$

Since $\delta(H) = 5$, it follows that $H = G'/E_1$,

where $\mathbf{E_1 \subseteq [T]_{G'}}$, and no edge in E_1 is incident with vertices of degree 13 or 4. While removing C from G, we have removed only two edges from [T, S] (two edges incident with one minor vertex in D').

$$\begin{aligned} \text{So} \quad s(G') &= s(G) - (1 \times 2) \\ &= 57 - 2 = 55 \end{aligned}$$

Then $s(H) \geq s(G') - 2|E_1|$.

Now, $|[S', T']| = s(H) - 2|[T']|$

$= 55 - 2(|E_1| + |[T']|)$

$\geq 55 - 2 \geq 53$ contradicting (2)

$\delta(H) \geq 6$

Now $n_5(H) = 0$

$$\begin{aligned} n_5(H) = 0 &\Rightarrow H \subseteq G'/v_1 && (\text{where } d(v_i) = 5) \\ &\Rightarrow |S'| \leq 11 - 4 = 7 && (\text{since } v_1 \text{ has atleast 4 major neighbours in } G') \\ &\Rightarrow \delta(H) \geq 8 && (\text{using VAL}) \qquad (3) \end{aligned}$$

Let u_1 be a vertex of degree 7 that has only major neighbours in G.

Then $d_{G'}(u_1) = 7$

Now by (3), $\delta(H) \geq 8$

$\delta(H) \geq 8 \Rightarrow H \subseteq G'/u_1$

$\Rightarrow |S'| \leq 11 - 7 = 4$

$\Rightarrow 11$

We note that $n_{10} + n_{11} = 0$ in G.

Hence $n''_8 + n''_9 \leq n_{10} + n_{11} + n_{12} + n_{13} + n_{14} + n_{15} = 11$

So, $|V(H)| \leq 11$ is a contradiction.

This completes the proof.

Lemma 6:

If G is an - critical graph with $\Delta = 15, \delta = 7$, $n = \Delta + 4, n_\Delta = \Delta - 4$ and $s(G) = 59$, then $2m \geq \Delta^2$.

Proof:

Assume the contrary that $2m < \Delta^2$. Then tha possible degree sequences are

i) $7^5\, 8^3\, 15^{11}$
ii) $7^6\, 8\, 9\, 15^{11}$
iii) $7^7\, 10\, 15^{11}$ given in lemma 4.

By **R[8]**, $|[T]| \leq 2$. But by VAL, $|[T]| \leq 1$

Now $|[S,T]| = \begin{cases} 59, & \text{if } |[T]| = 0 \\ 57, & \text{if } |[T]| = 1 \end{cases}$

Then in G, one of the following five cases arises:

i) $|[T]| = 0$, four major vertices have 6 minor neighbours and seven major vertices have 5 minor neighbours.
ii) $|[T]| = 0$, two major vertices have 6 minor neighbours and one major vertex have seven minor neighbours and 8 major vertices have 5 minor neighbours.
iii) $|[T]| = 0$, two major vertices have 7 minor neighbours and nine major vertices have 5 minor neighbours.
iv) $|[T]| = 1$, two major vertices have 6 minor neighbours and nine major vertices have 5 minor neighbours.
v) $|[T]| = 0$, one major vertex has 7 minor neighbours and two major vertices have 5 minor neighbours and 8 major vertices have 5 minor neighbours.

Also $d(v) \leq n_\Delta + 1 = 12$ for all $v \in T$.

Let $\pi(G) = \begin{cases} 7^5\, 8^3\, 15^{11} \\ 7^6\, 8 9 15^{11} \\ 7^7\, 10 15^{11} \end{cases}$

Let v_1 be a vertex of degree 7.

Let D be a subset of $T/\{v_1\}$ with $|D| = 6$.

Let $D' = T/D$. Then $|D'| = 2$ and $v_1 \in D'$.

Let $D = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ and $D' = \{v_1, v_2\}$.

We shall fix the degree of u_i's, v_i's accordingly.

Let $G_1 = G/D$.

Now $|V(G_1)| = 13$. Then $\Delta(G_1) \leq 12$.

Since $\delta(G) = 7$ and $|[T]| \leq 1$, we have $\delta(G_1) = 6$.

Let $V_1 = \{v \in V(G_1) : d_{G_1}(v) \geq 9\}$,

$V_2 = \{v \in V(G_1) : d_{G_1}(v) < 9\}$.

As in lemma 6, we can check that G_1 is hamiltonian.

Let c be a hamitonian cycle of G_1.

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Let $G' = G/E(C)$.
Since G is of class 2, G' is of class 2.
Let S', T' be defined as in lemma 6.
Then $|S'| \le 11$ and $\delta(H) \ge 4$.

δ(H)=5

Now $|S'| = 11$

Then $V(H) = V(G')$ and $d_H(v_1) = 5$..
Now we consider the cases (i) to (v).
Note that $d_H(v_1) = 5$. And so in H, V_1 has 4 major neighbourhood in case (i) and (iii) and has at least 3 major neighbourhood in cases (iv) and (v).

Then in (i) $|[S', T']| \le (4\times3) + (4\times6) + (3\times5) = 51$.

(ii) $|[S', T']| \le (4\times3) + (2\times6) + (1\times7) + (4\times5) = 51$,

(iii) $|[S', T']| \le (4\times3) + (2\times7) + (5\times5) = 51$,

(iv) $|[S', T']| \le (3\times3) + (2\times6) + (6\times5) = 51$,

(v) $|[S', T']| \le (3\times3) + (1\times7) + (7\times5) = 51$. (1)

But in all cases ,

$$s(G') = s(G) - (2\times2)$$
$$= 59 - 4 = 55$$

Also $\delta(H) = 5$ and so $H = G'/E_1$,where $E_1 \subseteq [T]_{G'}$, and no edge in E_1 is incident with vertices of degree 13 or 5 (Then $|E_1| \le 1$).

Now in $|[T']| + |E_1| = |[T]_{G'}| \le 1$

Then $s(H) = s(G') - 2|E_1|$
$= 55 - 2|E_1|$

Now in (i) – (iii), $H = G'$ and $|[T', S']| = 55$

In (iv) and (v) $|[T', S']| \ge 55 - 2 = 53$ contradicting to (1)

δ(H)≥6

Then $H \subseteq G'/v_1$ where $d(v_1)=5$

Now $n_5(H) = 0 \Rightarrow |S'| \le 11 - 3 = 8$ (since v_1 has at least 3 major nrighbours in G).

$\Rightarrow \delta(H) \ge 7$ (using VAL)

Now we have three possible degree sequences:

i) $7^6\ 8\ 9\ 15^{11}$
ii) $7^5\ 8^4\ 5^{11}$
iii) $7^7\ 10\ 15^{11}$

Let $d(v_2) = \begin{cases} 8 & \text{in (i) and (ii)} \\ 10 & \text{in (iii)} \end{cases}$

Then $d_{G'}(v_2) = \begin{cases} 6 & \text{in (i) and (ii)} \\ 8 & \text{in (iii)} \end{cases}$

Also by VAL, v_2 has at most one minor neighbours in G.

$$\delta(H) \geq 7 \Rightarrow H \subseteq G'/v_2$$

$$\Rightarrow |S'| \leq \begin{cases} 11-5 & \text{in (i) and (ii)} \\ 11-7 & \text{in (iii)} \end{cases}$$

$$\Rightarrow |S'| \leq \begin{cases} 6 & \text{in (i) and (ii)} \\ 4 & \text{in (iii)} \end{cases}$$

$$\Rightarrow \delta(H) \geq \begin{cases} 9 & \text{in (i) and (ii)} \\ 11 & \text{in (iii)} \end{cases}$$

$$\Rightarrow |\delta(H)| \leq 11, \quad \text{a contradiction.}$$

This completes the proof.

Theorem 1:

If G is a 15-critical graph with $\delta = 7$ and $n_\Delta = \Delta - 4$, then $2m \geq \Delta^2$.

Proof:

By Lemma 5 and Lemma 6, we get the result.

Theorem 2:

If G is a 15-critical graph with $\delta = 7$ and $n_\Delta = \Delta - 5$, then $2m \geq \Delta^2$.

Proof:

By R11, $55 \leq s(G) < 60$ and s(G) is odd and so s(G) is 55 or 57 or 59.

Since $n_\Delta = (\Delta - 5, n_\Delta = 11)$ is odd and also s(G) is odd, in the possible degree sequences the number of odd vertices is odd. It is impossible.

Hence the theorem.

Proof of the Main Theorem:

Theorem 3:

If G is a 15-critical graph with $\delta = 7$ then $2m \geq \Delta^2$.

Proof:

By VAL, $n_\Delta \geq \Delta - 5 + 2 \geq \Delta - 5$.

By R6, R7 and lemma 1, it is enough to verify the result when n $n_\Delta \geq \Delta + 4$ and $n_\Delta = \Delta - 4$ and $\Delta - 5$.

By Theorem 1 and 2, the main theorem follows.

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