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A STUDY ON T-FUZZY ℓ-IDEALS IN LATTICE ORDERED GROUP

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Abstract:
A lattice ordered group (ℓ-group) is a non-empty set G with binary operation + and binary relation ≤ such that (G, +) is a group and (G, ≤) is a lattice. In this paper, we introduced the triangular norm (t-norm) on fuzzy ideals of an ℓ-group and some related results are investigated. We also discussed about an ℓ-homomorphism on T-fuzzy ℓ-ideals.

Key Words: ℓ-group, ℓ-homomorphism, Fuzzy ℓ-group, Fuzzy ℓ-ideal, t-norm & T-fuzzy ℓ-ideal

1. Introduction:

The concept of fuzzy sets was introduced by L. A. Zadeh [18] and it has been developed by several algebraists in many directions [7, 8, 9, 10, 13]. Rosenfeld [14] and Das [4] investigated the theory of fuzzy groups. N. Ajmal and K. V. Thomas [1] analyzed fuzzy lattices. The properties of lattices were studied in [5, 6]. Natarajan and Vimala [11, 12] introduced the theory of ideals in ℓ-groups and they presented many prominent results. G. S. V. Sathyasaibaba [15] studied the concept of fuzzy lattice ordered groups as a mapping from ℓ-group into a complete lattice. He introduced L-fuzzy ℓ-ideals and proved some related results. Bharathi and Vimala [3] defined fuzzy ℓ-ideals of ℓ-group in new notion and they studied distributivity of fuzzy ℓ-ideals. Also, Vimala [17] investigated on homomorphism of fuzzy ℓ-ideals. Triangular norms (t-norms) were introduced by Schweizer and Sklar [16] due to the development of metric space theory. Now a day, t-norms contribute many applications in all fields of mathematics. In this paper, we introduced the concept of t-norms on fuzzy ℓ-ideals of ℓ-group. Also some properties of T-fuzzy ℓ-ideals are considered. In section II, we review some elementary definitions and results which are used to understand this paper. In section III, the notion of T-fuzzy ℓ-ideals is introduced and some related results are derived.

2. Preliminaries:

In this section, we have presented the basic definitions and results of ℓ-groups, ℓ-ideals, ℓ-homomorphism, fuzzy sets and fuzzy ℓ-groups which are useful for subsequent discussions.

Definition 2.1 [2] A non-empty set G is called a lattice ordered group (ℓ-group) iff
- (i) (G,+) is a group
- (ii) (G, ≤) is a lattice
- (iii) $x \le y$ implies $a+x+b \le a+y+b$ for all x,y,a,b in G.

Definition 2.2 [2] Let G be an ℓ-group. A non-empty subset I of G is called an ℓ-ideal of G if
- (i) I is a subgroup of G.
- (ii) I is a sublattice of G.

Definition 2.3[2] Let G, G' be two ℓ-groups. A function f : G → G' is called an ℓ-homomorphism if (i) $f(x \wedge y) = f(x) \wedge f(y)$
(ii) $f(x \vee y) = f(x) \vee f(y)$.
(iii) $f(x + y) = f(x) + f(y)$ for all $x, y \in G$.

Definition 2.4 [19] A fuzzy set of a non-empty set X is a function μ: X → [0,1].

Definition 2.5 [19] Let X be any non-empty set and μ be a fuzzy set defined on X and t ∈ [0, 1]. Then the set $\mu_t = \{x \in X / \mu(x) \ge t\}$ is called the level set of μ.

Definition 2.6 [19] Let X be any non-empty set and μ be a fuzzy set defined on X. Then the set $\{ \mu(x) / x \in X\}$ is called the image of μ and is denoted by Im(μ)

Definition 2.7 [19] Let X be any non-empty set and μ be a fuzzy set defined on X. The set $\{x / x \in X \text{ and } \mu(x) \ge 0\}$ is called the support of μ and it is denoted by supp(μ).

Definition 2.8 [15] Let G= (G, +, ∧, ∨) be an ℓ-group. A fuzzy set μ defined on G is said to be fuzzy lattice ordered group (fuzzy ℓ-group) of G if
- (i) $\mu(x+y) \ge \mu(x) \wedge \mu(y)$
- (ii) $\mu(-x) = \mu(x)$
- (iii) $\mu(x \vee y) \ge \mu(x) \wedge \mu(y)$
- (iv) $\mu(x \wedge y) \ge \mu(x) \wedge \mu(y)$ for all x,y in G.

Theorem 2.9 [15] Let G be an ℓ-group and μ be a fuzzy lattice ordered group. Then $\mu(0) \ge \mu(x)$ for all x ∈G.

Definition 2.10 [16] A t-norm is a function T : [0,1] X [0,1] → [0,1] that satisfies the following conditions for all x, y, z ∈ [0,1];
- (i) T(x, 1) = x
- (ii) T(x, y) = T(y, x)
- (iii) T(x, T(y, z)) = T(T(x,y), z)
- (iv) T(x,y) ≤ T(x, z) whenever y ≤ z.

Theorem 2.11 [17] Let G_1, G_2 be two ℓ-groups and f : $G_1 \to G_2$ be the ℓ-homomorphism. Let $\mu_1$be the fuzzy ideal of G_1 and μ_2 be the fuzzy ideal of G_2. Then the pre-image of f is defined by $[f^{-1}(\mu_2)](x) = \mu_2[f(x)]$ for $x \in G_1$.

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3. T-Fuzzy ℓ-Ideals in ℓ-Group:

Definition 3.1 A fuzzy ℓ-ideal μ of an ℓ-group G is called T-fuzzy ℓ-ideal of G if

(i) $\mu(x-y) \geq T(\mu(x), \mu(y))$

(ii) $\mu(x \vee y) \geq T(\mu(x), \mu(y))$

(iii) $\mu(x \wedge y) \geq T(\mu(x), \mu(y))$ for all x,y∈ G.

Example 3.2 Consider the ℓ-group (G, +, ∨, ∧) such that G = { (x,y)/ x, y ∈Z**}.** Define a fuzzy ℓ-ideal μ on G such that μ(x) = $\begin{Bmatrix} 0.5 \text{ if } y = 0 \\ 0 \text{ if } y \neq 0 \end{Bmatrix}$ and define a t-norm T such that T(x, y) = max(x+y-1, 0). Then μ is a T-fuzzy ℓ-ideal of G.

Definition 3.3 A fuzzy ℓ-group μ of an ℓ-group G is said to be T- fuzzy ℓ-group if

(i) $\mu(x-y) \geq T(\mu(x), \mu(y))$

(ii) $\mu(x \vee y) \geq T(\mu(x), \mu(y))$

(iii) $\mu(x \wedge y) \geq T(\mu(x), \mu(y))$ for all x,y∈ G.

Now we state the following theorems without proof as they are obvious.

Theorem 3.4 Every T-fuzzy ℓ-ideal of an ℓ-group G is a T-fuzzy ℓ-group of G. But the converse need not be true.

Theorem 3.5 A fuzzy set μ of G is the T-fuzzy ℓ-ideal of G if and only if the level set μ_t = { x ∈ G / $\mu(x) \geq t$} is an ℓ-ideal of G when it is non-empty.

Then we have the following theorems related to ℓ-homomorphism on T-fuzzy ℓ-ideals.

Theorem 3.6 Let G_1, G_2 be two ℓ-groups and f : $G_1 \to G_2$ be the ℓ-homomorphism. Let μ_1 be the T-fuzzy ℓ-ideal of G_1. Then there exist a T-fuzzy ℓ-ideal μ_2 of G_2 such that $\mu_2[f(x)]=\mu_1(x)$ for all x ∈ G.

Proof:

For any $x_1, x_2 \in G_1$

(i) $\mu_2[f(x_1) - f(x_2)] = \mu_2[f(x_1 - x_2)]$
$= \mu_1(x_1 - x_2)$.
$\geq T(\mu_1(x_1), \mu_1(x_2))$
$= T(\mu_2(f(x_1)), \mu_2(f(x_2)))$

$\Rightarrow \mu_2[f(x_1) - f(x_2)] \geq T(\mu_2(f(x_1)), \mu_2(f(x_2)))$

(ii) $\mu_2[f(x_1) \wedge f(x_2)] = \mu_2[f(x_1 \wedge x_2)]$
$= \mu_1(x_1 \wedge x_2)$.
$\geq T(\mu_1(x_1), \mu_1(x_2))$
$= T(\mu_2(f(x_1)), \mu_2(f(x_2)))$

$\Rightarrow \mu_2[f(x_1) \wedge f(x_2)] \geq T(\mu_2(f(x_1)), \mu_2(f(x_2)))$

(iii) $\mu_2[f(x_1) \vee f(x_2)] = \mu_2[f(x_1 \vee x_2)]$
$= \mu_1(x_1 \vee x_2)$.
$\geq T(\mu_1(x_1), \mu_1(x_2))$
$= T(\mu_2(f(x_1)), \mu_2(f(x_2)))$

$\Rightarrow \mu_2[f(x_1) \vee f(x_2)] \geq T(\mu_2(f(x_1)), \mu_2(f(x_2)))$

Hence μ_2 is a T-fuzzy ℓ-ideal of G_2.

Theorem 3.7 Let G_1, G_2 be two ℓ-groups and f : $G_1 \to G_2$ be the ℓ-homomorphism. Let μ_1 be the T-fuzzy ℓ-ideal of G_1 and μ_2 be the T-fuzzy ℓ-ideal of G_2. Then the pre-image of f defined by $[f^{-1}(\mu_2)](x) = \mu_2[f(x)]$ is a T-fuzzy ℓ-ideal of G_1.

Proof:

For any x, y∈ G_1

(i) $[f^{-1}(\mu_2)](x - y) = \mu_2[f(x - y)]$
$= \mu_2[f(x) - f(y)]$
$\geq T(\mu_2[f(x)], \mu_2 f(y)])$
$= T(f^{-1}(\mu_2)(x), f^{-1}(\mu_2)(y))$

$\Rightarrow [f^{-1}(\mu_2)](x - y) \geq T(f^{-1}(\mu_2)(x), f^{-1}(\mu_2)(y))$

(ii) $f^{-1}(\mu_2)](x \wedge y) = \mu_2[f(x \wedge y)]$
$= \mu_2[f(x) \wedge f(y)]$
$\geq T(\mu_2[f(x)], \mu_2 f(y)])$
$= T(f^{-1}(\mu_2)(x), f^{-1}(\mu_2)(y))$

$\Rightarrow [f^{-1}(\mu_2)](x \wedge y) \geq T(f^{-1}(\mu_2)(x), f^{-1}(\mu_2)(y))$

(iii) $f^{-1}(\mu_2)](x \vee y) = \mu_2[f(x \vee y)]$
$= \mu_2[f(x) \vee f(y)]$
$\geq T(\mu_2[f(x)], \mu_2 f(y)])$
$= T(f^{-1}(\mu_2)(x), f^{-1}(\mu_2)(y))$

$\Rightarrow [f^{-1}(\mu_2)](x \vee y) \geq T(f^{-1}(\mu_2)(x), f^{-1}(\mu_2)(y))$

Thus the pre-image of f is a T-fuzzy ℓ-ideal of G_1.

Theorem 3.8 Let μ be the T-fuzzy ℓ-ideal of an ℓ-group G. Then the fuzzy set $\varphi : \frac{G}{\mu} \to [0,1]$ defined by $\varphi(g+\mu) = \mu(g)$ is the T-fuzzy ℓ-ideal of $\frac{G}{\mu}$.

Proof:

Let $g_1, g_2 \in G$.

(i) $\varphi[(g_1+\mu)-(g_2+\mu)] = \varphi[(g_1-g_2)+\mu]$
$= \mu(g_1-g_2)$
$\geq T(\mu(g_1), \mu(g_1))$
$= T(\varphi(g_1+\mu), \varphi(g_2+\mu))$

$\Rightarrow \varphi[(g_1+\mu)-(g_2+\mu)] \geq T(\varphi(g_1+\mu), \varphi(g_2+\mu))$

(ii) $\varphi[(g_1+\mu)\vee(g_2+\mu)] = \varphi[(g_1\vee g_2)+\mu]$
$= \mu(g_1\vee g_2)$
$\geq T(\mu(g_1), \mu(g_1))$
$= T(\varphi(g_1+\mu), \varphi(g_2+\mu))$

$\Rightarrow \varphi[(g_1+\mu)\vee(g_2+\mu)] \geq T(\varphi(g_1+\mu), \varphi(g_2+\mu))$

(iii) $\varphi[(g_1+\mu)\wedge(g_2+\mu)] = \varphi[(g_1\wedge g_2)+\mu]$
$= \mu(g_1\wedge g_2)$
$\geq T(\mu(g_1), \mu(g_1))$
$= T(\varphi(g_1+\mu), \varphi(g_2+\mu))$

$\Rightarrow \varphi[(g_1+\mu)\wedge(g_2+\mu)] \geq T(\varphi(g_1+\mu), \varphi(g_2+\mu))$

Theorem 3.9 Let G be an ℓ-group and I be an ℓ-ideal of G. If μ is a T-fuzzy ℓ-ideal of G/I then there exists a T-fuzzy ℓ-ideal φ of G such that φ_t= I for t= $\mu(0)$.

Proof:

Let x, y ∈ G.

(i) $\varphi(x-y) = \mu((x-y)+I)$
$= \mu((x+I)-(y+I))$
$\geq T(\mu((x+I), (y+I)))$
$= T(\varphi(x), \varphi(y))$
$\Rightarrow \varphi(x-y) \geq T(\varphi(x), \varphi(y))$

(ii) $\varphi(x\vee y) = \mu((x\vee y)+I)$
$= \mu((x+I)\vee(y+I))$
$\geq T(\mu((x+I), (y+I)))$
$= T(\varphi(x), \varphi(y))$
$\Rightarrow \varphi(x\vee y) \geq T(\varphi(x), \varphi(y))$

(iii) $\varphi(x\wedge y) = \mu((x\wedge y)+I)$
$= \mu((x+I)\wedge(y+I))$
$\geq T(\mu((x+I), (y+I)))$
$= T(\varphi(x), \varphi(y))$
$\Rightarrow \varphi(x\wedge y) \geq T(\varphi(x), \varphi(y))$

Let $x \in \varphi_t$ for t=μ(0)
$\Leftrightarrow \varphi(x) = (0)$
$\Leftrightarrow \mu(x+I) = \mu(0+I) = \mu(I)$
$\Leftrightarrow x \in I$.
Hence φ_t= I for t= $\mu(0)$..

Conclusion:

In this paper, the concept of T-fuzzy ℓ-ideals has been inspected. This work absorbed on some properties of T-fuzzy ℓ-ideals of an ℓ-group .In future, we can extend this notion to T-fuzzy prime ideals, T-fuzzy quotient ideals, etc.

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