ISSN: 2456 – 3080
International Journal of Applied and Advanced Scientific Research
Impact Factor 5.255, Special Issue, February - 2017
International Conference on Advances in Theoretical and Applied Mathematics – ICATAM 2017
On 14th February 2017 Organized By
Madurai Sivakasi Nadars Pioneer Meenakshi Women's College, Poovanthi, Tamilnadu

A NOTE ON RADIAL RADIO NUMBER OF A GRAPH

Selvam Avadayappan*, M. Bhuvaneshwari & S.Vimalajenifer*****
Research Department of Mathematics, Virudhunagar Hindu Nadars' Senthikumara Nadar College, Virudhunagar, Tamilnadu

Cite This Article: Selvam Avadayappan, M. Bhuvaneshwari & S.Vimalajenifer, "A Note on Radial Radio Number of a Graph", International Journal of Applied and Advanced Scientific Research, Special Issue, February, Page Number 62-68, 2017.

Abstract:

Let G be any simple, connected, undirected and finite graph. A radial radio labeling, f, of G is a function $f:V(G)\to\{1,2,...\}$ satisfying the condition $d(u,v)+|f(u)-f(v)|\geq 1+rad(G)$, for all $u,v\in V(G)$, where $d(u,v)$ and $rad(G)$ denote the distance between the vertices u and v and the radius of the graph G, respectively. The span of a radial radio labeling f is the largest integer in the range of f and is denoted by $span(f)$. The radial radio number of G, $rr(G)$, is the minimum span taken over all radial radio labelings of G. Denote $N_0(v)=\{v\}$, $N_\lambda(v)=\{w:d(v,w)=\lambda\}$ and $\mathcal{N}_\lambda[v]=\bigcup_{i=0}^{\lambda}N_i(v)$. In this paper, we introduce some new concepts related with the sequence of the radial radio numbers of the induced subgraphs induced by $\mathcal{N}_\lambda[v]$.

Key Words: Frequency Assignment Problem, Radius, Diameter, Radio Labeling, Radial Radio Labeling, Radio Number, Radial Radio Number & rr Sequence.

1. Introduction:

In this paper, we consider only simple, connected, undirected and finite graphs. For basic notations and terminology, we follow [2]. Let $G=(V,E)$ be a simple connected graph. A vertex $v\in V$ is called a *full vertex*, if it is adjacent to all other vertices. Let $u,v\in V$ be any two vertices of G. The *distance* $d(u,v)$ between u and v, is the length of a shortest $(u,v)-$path in G. For any vertex $u\in V$, the *eccentricity*, $e(u)$, of u is the distance of a vertex farthest from u. The *radius* of a graph G is the minimum eccentricity among all the vertices and is denoted by $rad(G)$. The *diameter* of G is the maximum eccentricity among all the vertices and is denoted by $diam(G)$. The relation between $rad(G)$ and $diam(G)$ is given by the inequality $rad(G)\leq diam(G)\leq 2rad(G)$ [6]. For further details on distance in graphs, one can refer [3].

For a subset S of V, let $<S>$ denote the induced subgraph of G induced by S. A *clique* C is a subset of V with maximum number of vertices such that $<C>$ is complete. The *clique number* of a graph G, denoted by ω, is the number of vertices in a clique of G.

For any vertex $v\in V(G)$, the *open neighborhood* $N(v)$ is the set of all vertices adjacent to v. That is, $N(v)=\{u\in V(G):uv\in E(G)\}$. The *closed neighborhood* $N[v]$ of v is defined by $N[v]=N(v)\cup\{v\}$. Denote $N_0(v)=\{v\}$, $N_\lambda(v)=\{w:d(v,w)=\lambda\}$ and $\mathcal{N}_\lambda[v]=\bigcup_{i=0}^{\lambda}N_i(v)$.

In 1960's Rosa[11] introduced the concept of graph labeling. A *graph labeling* is an assignment of numbers to the vertices or edges or both, satisfying some constraint. Rosa named the labeling introduced by him as $\beta-valuation$ and later on it becomes a very famous interesting graph labeling called *graceful labeling*, which is the origin for any graph labeling problem. Motivated by real life problems, many mathematicians introduced various labeling concepts[8]. Here, we see one of the familiar graph labelings in graph theory.

The problem of assigning frequencies to the channels for the FM radio stations is known as *Frequency Assignment Problem* (FAP). This problem was studied by W. K. Hale[9]. In a telecommunication system, the assignment of channels to FM radio stations play a vital role. Motivated by the FAP, Chartrand et al.,[4] introduced the concept of radio labeling. For a given k, $1\leq k\leq diam(G)$, a *radio k- coloring*, f, is an assignment of positive integers to the vertices satisfying the following condition:

$$d(u,v)+|f(u)-f(v)|\geq 1+k \qquad (1)$$

for all $u,v\in V$. Whenever, $diam(G)=k$, the radio k- coloring is called a *radio labeling* [5] of G. The *span* of a radio labeling f is the largest integer in the range of f and is denoted by $span(f)$. The *radio number* of G is the minimum span taken over all radio labelings of G and is denoted by $rn(G)$.

Motivated by the work of Chartrand et al., on radio labeling, KM. Kathiresan and S. Vimalajenifer[10] introduced the concept of radial radio labeling. A *radial radio labeling* f of G is a function $f:V\to\{1,2,...\}$ satisfying the condition,

$$d(u,v)+|f(u)-f(v)|\geq 1+rad(G) \qquad (2)$$

ISSN: 2456 – 3080
International Journal of Applied and Advanced Scientific Research
Impact Factor 5.255, Special Issue, February - 2017
International Conference on Advances in Theoretical and Applied Mathematics – ICATAM 2017
On 14th February 2017 Organized By
Madurai Sivakasi Nadars Pioneer Meenakshi Women's College, Poovanthi, Tamilnadu

for all $u, v \in V$. This condition is obtained by taking $k = rad(G)$ in (1). The above condition is known as *radial radio condition*. The *span* of a radial radio labeling f is the largest integer in the range of f. The *radial radio number* is the minimum span taken over all radial radio labelings of G and is denoted by $rr(G)$. That is, $rr(G) = \min_{f} \max_{v \in V} f(v)$, where the minimum runs over all radial radio labelings of G.

The *centre* is the subgraph of G induced by the set of vertices of minimum eccentricity. Any graph G which is isomorphic to its centre is called a *self centered graph*. Note that in a self centered graph G, $rad(G) = diam(G)$.

Theorem 1.1: *Let G be a self centered graph on n vertices. Then $rr(G) \geq n$.*

Proof: Since G is self centered, $rad(G) = diam(G)$. This implies that, $rn(G) = rr(G)$. But it is already proved that, $rn(G) \geq n$ [7]. Thus we have, $rr(G) \geq n$.

Fact (1): Two vertices u and v have the same labels only when $d(u,v) \geq 1 + rad(G)$.

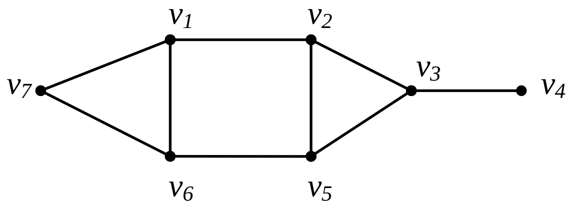
For, let G be a graph on n vertices and let f be a radial radio labeling of G. Then the radial radio condition for G give $d(u,v) + |f(u) - f(v)| \geq 1 + rad(G)$, for all $u, v \in V(G)$. This implies that, $|f(u) - f(v)| \geq 1 + rad(G) - d(u,v)$, for all $u, v \in V(G)$. Therefore, if $f(u) = f(v)$, then $d(u,v) \geq 1 + rad(G)$. In this paper, we introduce a new concept $(\mu_\lambda(v)) - rr$ sequence of a graph, where $\mu_\lambda(v) = rr(<\mathcal{N}_\lambda[v]>)$, $v \in V(G)$. In addition, we mainly discuss about the $(\mu_1(v)) - rr$ sequence of a graph. Also, the regularity of $(\mu_1(v)) - rr$ sequence has been studied.

2. The rr Sequence of a Graph:

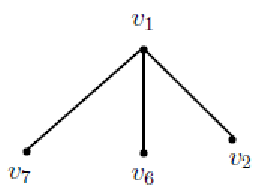
We have already studied about many sequences such as degree sequence, graphic sequence and so on. Likewise here, we introduce a new sequence based on the radial radio number of a graph.

For a vertex $v \in V(G)$, let $\mu_\lambda(v)$ denote the radial radio number of the induced subgraph induced by $\mathcal{N}_\lambda[v]$. That is, $\mu_\lambda(v) = rr(<\mathcal{N}_\lambda[v]>)$. Then the sequence $(\mu_\lambda(v))_{v \in V(G)}$ arranged in decreasing order is called the $(\mu_\lambda(v)) - rr$ *sequence* of the graph G. The following example gives the $(\mu_1(v)) - rr$ sequence of the graph G.

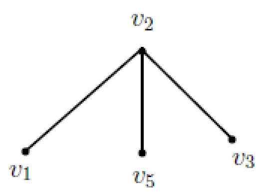
Example 2.1: *Consider the graph G shown in Figure 2.1. The subgraphs induced by $\mathcal{N}_1[v]$, $v \in V(G)$ are also given in Figure 2.1.*



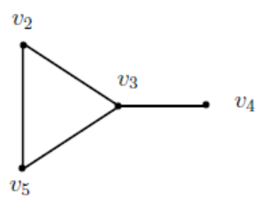
G



□□[v_1]

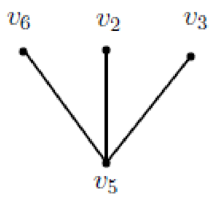


□ $_1[v_2]$



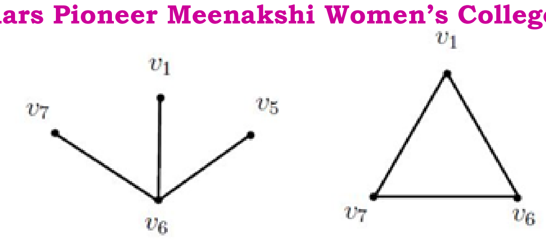
□ $_1[v_3]$

□ $_1[v_4]$



□ $_1[v_5]$

ISSN: 2456 – 3080
International Journal of Applied and Advanced Scientific Research
Impact Factor 5.255, Special Issue, February - 2017
International Conference on Advances in Theoretical and Applied Mathematics – ICATAM 2017
On 14th February 2017 Organized By
Madurai Sivakasi Nadars Pioneer Meenakshi Women's College, Poovanthi, Tamilnadu



□ $_1[v_6]$ □ $_1[v_7]$

Figure 2.1

Here, $rr(<\mathcal{N}_1[v_i]>) = 2, i = 1,2,4,5,6$ and $rr(<\mathcal{N}_1[v_i]>) = 3, i = 3,7$. Hence the $(\mu_1(v)) - rr$ sequence of G is $(3,3,2,2,2,2,2)$.

In this paper, we mainly discuss about the $(\mu_1(v)) - rr$ sequence of a graph.

A graph G is said to be $\mu - rr$ *regular* if the $(\mu_1(v)) - rr$ sequence of G is $(\mu, \mu, ..., \mu)$, where $\mu = rr(<\mathcal{N}_1[u]>)$, for all $u \in V(G)$ and μ is called $rr-$ *constant*.

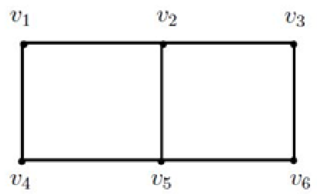


Figure 2.2

For the graph shown in Figure 2.2, $(\mu_1(v)) - rr$ sequence is $(2,2,2,2,2,2)$ and hence the graph is $2 - rr$ regular.

A $\mu - rr$ regular graph G is said to be $\mu - rr$ *highly regular* if $rr(G) = \mu$.

For example, the complete graph K_n is $n - rr$ highly regular and the star graph $K_{1,n}$ is $2 - rr$ highly regular. Wheel graph W_n is $3 - rr$ highly regular, when n is even. Any tree other than $K_{1,n}$ is $2 - rr$ regular but not $2 - rr$ highly regular. The complete bipartite graph $K_{m,n} (m \geq n \geq 2)$ is $2 - rr$ regular but not $2 - rr$ highly regular.

A graph G on n vertices is said to be rr *irregular*, if it is not $\mu - rr$ regular, for any μ, $2 \leq \mu \leq n$.

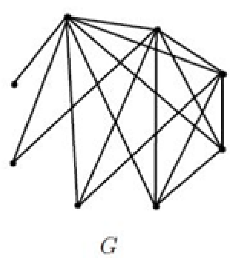


Figure 2.3

For the graph G, shown in Figure 2.3, the $(\mu_1(v)) - rr$ sequence is $(5,5,5,5,5, 4,3,2)$ and hence the graph is rr irregular.

3. Existence of $\mu - rr$ Regular and $\mu - rr$ Highly Regular Graphs:

In this section, we construct graphs with given integer μ as $rr-$ constant and graphs which are $\mu - rr$ regular but not $\mu - rr$ highly regular.

Theorem 3.1: *For any given $n \geq 4$, there exists a $n - rr$ regular graph G but not $n - rr$ highly regular.*

Proof: The required graph G_n is constructed using the three copies K_{n-2}, $n \geq 4$. Take

$$V(G_n) = \{v, v_1, v_2, ..., v_{n-2}, w, w_1, w_2, ..., w_{n-2}, u, u_1, u_2, ..., u_{n-2}\} \quad and$$

$$E(G_n) = \{vw, wu, uv\} \cup \{vv_i, wv_i : 1 \leq i \leq n-2\} \cup \{v_i v_j : 1 \leq i \neq j \leq n-2\}$$
$$\cup \{ww_i, uw_i : 1 \leq i \leq n-2\} \cup \{w_i w_j : 1 \leq i \neq j \leq n-2\}$$

ISSN: 2456 – 3080
International Journal of Applied and Advanced Scientific Research
Impact Factor 5.255, Special Issue, February - 2017
International Conference on Advances in Theoretical and Applied Mathematics – ICATAM 2017
On 14th February 2017 Organized By
Madurai Sivakasi Nadars Pioneer Meenakshi Women's College, Poovanthi, Tamilnadu

$\cup \{uu_i, vu_i : 1 \le i \le n-2\} \cup \{u_i u_j : 1 \le i \ne j \le n-2\}$.

We have $|V(G_n)| = 3n-3$ and $rad(G) = 2$. The graph G_4 is shown in Figure 3.1.

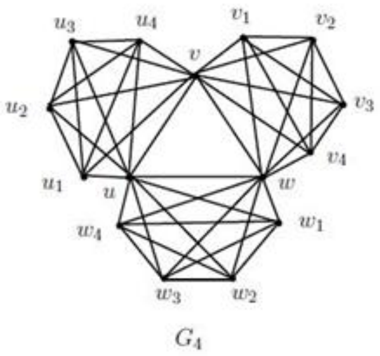


Figure 3.1

Now, we have to show that $rr(G_n) \ne rr(< N[u] >)$, for any $u \in V(G_n)$.

For $n \ge 6$, define $f : V(G_n) \to \{1,2,3,...\}$ such that

$f(v) = 1$;
$f(v_1) = 4$;
$f(v_2) = 7$;
$f(v_i) = f(v_{i-1}) + 2$, $3 \le i \le n-2$;
$f(w) = f(v_{n-2}) + 2$;
$f(w_1) = 2$;
$f(w_2) = 5$;
$f(w_3) = 8$;
$f(w_i) = f(w_{i-1}) + 2$, $4 \le i \le \lceil \frac{n-2}{2} \rceil + 1$;
$f(w_{\lceil (n-2)/2 \rceil + 2}) = f(w) + 2$;
$f(w_i) = f(w_{i-1}) + 2$, $\lceil \frac{n-2}{2} \rceil + 3 \le i \le n-2$;
$f(u) = f(w_{\lceil (n-2)/2 \rceil + 1}) + 2$;
$f(u_1) = f(u) + 2$;
$f(u_i) = f(u_{i-1}) + 2$, $2 \le i \le n-4$;
$f(u_{n-3}) = 6$;
$f(u_{n-2}) = 3$.

The radial radio condition for G_n is

$$d(u,v) + |f(u) - f(v)| \ge 3 \quad (3)$$

for all $u, v \in V(G_n)$. Now, we have to show that f is a radial radio labeling for G_n. To prove this, it is enough to show that every pair of vertices satisfies (3).

Case 1: Consider the pair (u,v). Since $d(u,v) = 1$, we have $d(u,v) + |f(u) - f(v)| = 1 + |1 - (f(w_{\lceil (n-2)/2 \rceil + 1}) + 2)| \ge 3$. Thus the pair (u,v) satisfies (3). Similarly, the pairs (v,w) and (w,u) satisfy (3).

Case 2: Consider the pair (v, v_i), $1 \le i \le n-2$. Since $d(v, v_i) = 1$, $d(v, v_i) + |f(v) - f(v_i)| \ge 1 + |1 - 4| > 3$. Therefore, the pair (v, v_i), $1 \le i \le n-2$ satisfies (3). Similarly, the pairs (u, u_i) and (w, w_i), $1 \le i \le n-2$ satisfy (3).

Case 3: Consider the pair (v, w_i), $1 \le i \le n-2$. We have $d(v, w_i) = 2$, $d(v, w_i) + |f(v) - f(w_i)| \ge 2 + |1 - 2| \ge 3$. Thus

ISSN: 2456 – 3080
International Journal of Applied and Advanced Scientific Research
Impact Factor 5.255, Special Issue, February - 2017
International Conference on Advances in Theoretical and Applied Mathematics – ICATAM 2017
On 14th February 2017 Organized By
Madurai Sivakasi Nadars Pioneer Meenakshi Women's College, Poovanthi, Tamilnadu

the pair (v, w_i), $1 \le i \le n-2$ satisfies (3). Similarly, the pairs (w, u_i) and (u, v_i), $1 \le i \le n-2$ satisfy (3).

Case 4: Consider the pair (v, u_i), $1 \le i \le n-2$. Since $d(v, u_i) = 1$, we have $d(v, u_i) + |f(v) - f(u_i)| \ge 1 + |1 - (f(w_{\lceil (n-2)/2 \rceil + 1}) + 4)| > 3$ and so the pair (v, u_i), $1 \le i \le n-2$ satisfies (3). Similarly, the pairs (u, w_i) and (w, v_i), $1 \le i \le n-2$ satisfy (3).

Case 5: Consider the pair (v_i, v_j), $1 \le i \ne j \le n-2$. Since $d(v_i, v_j) = 1$ and the label difference between v_i and v_j, $1 \le i \ne j \le n-2$ is at least 2, the pair (v_i, v_j), $1 \le i \ne j \le n-2$ satisfies (3). Similarly, we can show that the pairs (u_i, u_j), (w_i, w_j), $1 \le i \ne j \le n-2$ satisfy (3).

Case 6: Consider the pair (v_i, w_j), $1 \le i, j \le n-2$. Since $d(v_i, w_j) = 2$, $1 \le i, j \le n-2$, $d(v_i, w_j) + |f(v_i) - f(w_j)| \ge 2 + |4 - 2| \ge 3$. Therefore, the pair (v_i, w_j), $1 \le i, j \le n-2$ satisfies (3). Similarly, we can prove that the pairs (u_i, w_j), (u_i, v_j), $1 \le i, j \le n-2$ satisfy (3).

Thus f is a radial radio labeling of G. Also, we have $span(f) = |V(G_n)| = 3n-3$, $n \ge 6$.

For $n = 4,5$, radial radio labeling of G_4 and G_5 are given in Figure 3.2.

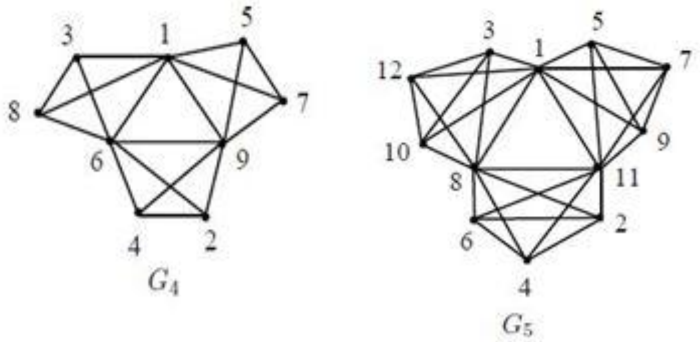


Figure 3.2

Hence $rr(G_n) \le 3n-3$.

Since $rad(G_n) = diam(G_n)$, G_n is self centered. Thus by Theorem 1.1, we have $rr(G_n) \ge 3n-3$ Hence, we can conclude that $rr(G_n) = 3n-3$.

Now, we find $rr(<\mathcal{N}_1[u]>)$, for all $u \in V(G_n)$. Here we have two types of vertices:

Type 1: vertex of degree $2n-2$.

Type 2: vertex of degree $n-1$.

Let $a \in V(G_n)$. If $deg(a) = 2n-2$, then $<\mathcal{N}_1[a]> \cong G$, where the graph G is given in Figure 3.3.

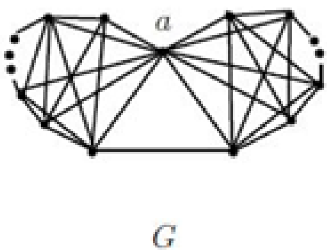


Figure 3.3

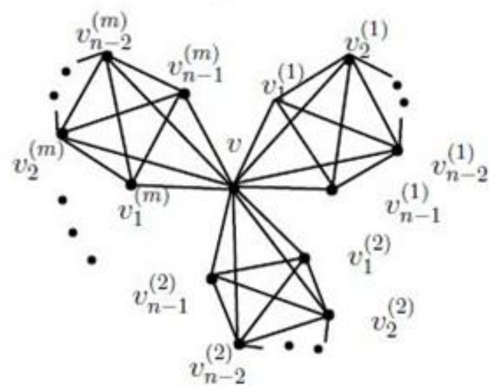
We have, $rr(G) = n$. Hence $rr(<\mathcal{N}_1[a]>) = n$. If $deg(a) = n-1$, then $<\mathcal{N}_1[a]> \cong K_n$. We have, $rr(K_n) = n$ [10] and so $rr(<\mathcal{N}_1[a]>) = n$. Hence $rr(<\mathcal{N}_1[u]>) = rr(<\mathcal{N}_1[v]>)$, for all $u, v \in V(G)$. Thus the $(\mu_1(v)) - rr$ sequence of G_n is $(n, n, ..., n)$. But $rr(G_n) = 3n-3$ and so $rr(G_n) \ne rr(<\mathcal{N}_1[u]>)$, for any

ISSN: 2456 – 3080
International Journal of Applied and Advanced Scientific Research
Impact Factor 5.255, Special Issue, February - 2017
International Conference on Advances in Theoretical and Applied Mathematics – ICATAM 2017
On 14th February 2017 Organized By
Madurai Sivakasi Nadars Pioneer Meenakshi Women's College, Poovanthi, Tamilnadu

$u \in V(G_n)$. Thus G_n is $n-rr$ regular graph but not $n-rr$ highly regular.

Theorem 3.2: *For any given integers* $m \geq 1$ *and* $n \geq 2$, *there exists a* $n-rr$ *highly regular graph of order* $m(n-1)+1$ *with* $rr-$ *constant* n.

Proof: Consider the complete graph K_n, $n \geq 2$. Let G be the required graph. Take $V(G) = \{v, v_1^{(1)}, v_2^{(1)}, ..., v_{n-1}^{(1)}, v_1^{(2)}, v_2^{(2)}, ..., v_{n-1}^{(2)}, ..., v_1^{(m)}, v_2^{(m)}, ..., v_{n-1}^{(m)},\}$ *and* $E(G) = \{vv_i^{(j)} : 1 \leq i \leq n-1, 1 \leq j \leq m\} \cup \{v_i^{(j)} v_s^{(j)} : 1 \leq i \neq s \leq n-1, 1 \leq j \leq m\}$. The graph G is shown in Figure 3.4.



G

Figure 3.4

Here, we have $\omega = n$ and $rad(G) = 1$.

Define $f : V(G) \to \{1,2,3,...\}$ by

$f(u_i^{(j)}) = i+1, \ 1 \leq i \leq n-1, 1 \leq j \leq m$;

$f(v) = 1$.

The radial radio condition for G is

$$d(u,v) + |f(u) - f(v)| \geq 2 \quad (4)$$

for all $u, v \in V(G)$. Now, we have to show that f is a radial radio labeling for G. Here, we have $d(v_i^{(s)}, v_j^{(t)}) = 2$, $1 \leq i \neq j \leq n-1$, $1 \leq s \neq t \leq m$. From the definition, $v_i^{(s)}$ and $v_j^{(t)}$, $1 \leq i \neq j \leq n-1$, $1 \leq s \neq t \leq m$ have the same labels. By Fact (1), it can be allowed and hence f satisfies (4). Thus f is a radial radio labeling of G. Also, $span(f) = n$. Thus $rr(G) \leq n$. It is already proven that $rr(G) \geq \omega$ [1] and hence $rr(G) \geq n$. Hence $rr(G) = n$.

Now, we find $rr(<\mathcal{N}_1[u]>)$, for all $u \in V(G)$. Here we have two types of vertices:

Type 1: vertex of degree $m(n-1)$.

Type 2: vertex of degree $n-1$.

Let $a \in V(G_n)$. If $deg(a) = m(n-1)$, then $<\mathcal{N}_1[a]> \cong G$. Hence $rr(<\mathcal{N}_1[a]>) = n$. If $deg(a) = n-1$, then $<\mathcal{N}_1[a]> \cong K_n$. We have, $rr(K_n) = n$ [10] and so $rr(<\mathcal{N}_1[a]>) = n$. Hence $rr(<\mathcal{N}_1[u]>) = rr(<\mathcal{N}_1[v]>)$, for all $u, v \in V(G)$. Thus the $(\mu_1(v)) - rr$ sequence of G is $(n, n, ..., n)$ and hence G is $n-rr$ regular. But $rr(G) = n$ and so $rr(G) = rr(<\mathcal{N}_1[u]>)$, for any $u \in V(G)$. Thus G is $n-rr$ highly regular.

Acknowledgment:

The third author thanks University Grants Commission, India for their financial support by Maulana Azad National Fellowship for Minority Students. (Grant no. F1-17.1/2015-16/MANF-2015-17-TAM-66646)

References:

1. Selvam Avadayappan, M. Bhuvaneshwari, S. Vimalajenifer, Radial Radio Number of a Graph (submitted).
2. R. Balakrishnan and K. Ranganathan, A Text Book of Graph Theory, Springer-Verlag, New York, Inc(1999).
3. F. Buckley and F. Harary, Distance in Graphs, Addison Wesley Reading, 1990.
4. G. Chartrand, D. Erwin, F. Harary and P. Zhang, Radio labeling of graphs, Bulletin of the institute of combinatorics and

its applications, vol. 33, pp. 77-85, 2001.
5. G. Chartrand and P. Zhang, Chromatic Graph Theory, CRC Press, Taylor and Francis Group, New York, 2009.
6. G. Chartrand and P. Zhang, Introduction to Graph Theory, McGraw Hill Publications(India) Private Limited, New Delhi, India, 2001.
7. C. Fernandance, A. Flores, M. Tomova and C. Wyles, The Radio Number of Gear Graphs, reprint available at front.math.ucdvis.edu/0809.2623.
8. J. A. Gallian, A Dynamic Survey of Graph Labeling, The Electronic Journal of Combinatorics, 2015.
9. W. K. Hale, Frequency Assignment: Theory and Applications, Proceeding of the IEEE, vol. 68, no. 12, pp. 1497-1514, 1980.
10. KM. Kathiresan and S. Vimlajenifer, Radial Radio Number of Graphs(submitted).
11. A. Rosa, On certain valuations of the vertices of a graph, Theory of Graphs (Internat. Symposium, Rome, July 1966), Gordon and Breach, N. Y. and Dunod Paris (1967) 349-355.