ISSN: 2456 – 3080
International Journal of Applied and Advanced Scientific Research
Impact Factor 5.255, Special Issue, February - 2017
International Conference on Advances in Theoretical and Applied Mathematics – ICATAM 2017
On 14th February 2017 Organized By
Madurai Sivakasi Nadars Pioneer Meenakshi Women's College, Poovanthi, Tamilnadu

PERFECT DEGREE SUPPORT IRREGULAR GRAPHS

M. Bhuvaneshwari* & Selvam Avadayappan**
Research Department of Mathematics, Virudhunagar Hindu Nadars' Senthikumara Nadar College, Virudhunagar, Tamilnadu

Cite This Article: M. Bhuvaneshwari & Selvam Avadayappan, "Perfect Degree Support Irregular Graphs", International Journal of Applied and Advanced Scientific Research, Special Issue, February, Page Number 21-24, 2017.

Abstract:

For a graph G(V,E), the support s(v) of a vertex v is defined as the sum of degrees of its neighbours. A graph G is said to be a k – perfect degree support graph (or simply a k – pds graph) if for any vertex v in G, the ratio of its support to its degree is the constant k. A graph G is called a (k, c) – linear degree support graph (or simply a (k,c) – lds graph) if, the support of any vertex exceeds k times of its degree by a constant c. In this paper, we study about the irregularity of k – pds graphs.

Key Words: Support of a Vertex, k – Perfect Degree Support Graph, (k,c) – Linear Degree Support Graph, Highly Irregular Graph & Neighbourly Irregular Graph

1. Introduction:

Throughout this paper we consider only finite, simple, undirected and connected graphs. For notations and terminology we follow [7]. A graph G is said to be *r- regular*, if every vertex of G has degree r. For a ≠ b, a graph G is said to be *(a, b) – biregular* if the degree d(v) is either a or b for any vertex v in G. The *support* s(v) of a vertex v is the sum of degrees of its neighbours. That is, $s(v) = \sum_{u\in N(v)} d(u)$. Note that the support of any vertex in an r – regular graph is r^2. In a graph *G*, deleting an edge *uv* and introducing a new vertex *w* together with the new edges *uw* and *vw* is called the *subdivision of the edge uv.* The *subdivision graph*, $S_1(G)$, of a graph G is obtained from the graph *G* by subdividing every edge of *G* exactly once. For example, the graph $S_1(K_{1,5})$ is shown in Figure 1.

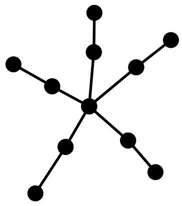


Figure 1: $S_1(K_{1,5})$

A vertex v is said to be a *k - regular adjacency vertex* (or simply a k - RA vertex) if d(u) = k for all u ∈ N(v). An *RA vertex* is a vertex which is a k – RA vertex for some k ≥ 1. A graph G in which every vertex is an RA vertex is said to be an *RA graph.* Obviously regular graphs and complete bipartite graphs are RA graphs. It has been proved that,

Result A [2] Any connected RA graph is either a regular graph or a biregular bipartite graph with each partition having vertices of same degree.

A connected graph G is said to be a *highly irregular graph* (or simply a *HI graph*), if each of its vertices is adjacent only to vertices with distinct degrees. For example, the graphs shown in Figure 2 are highly irregular.

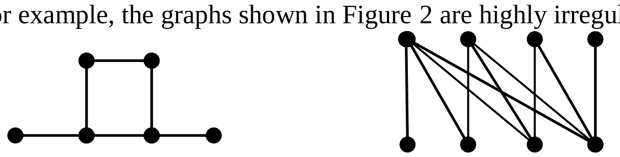
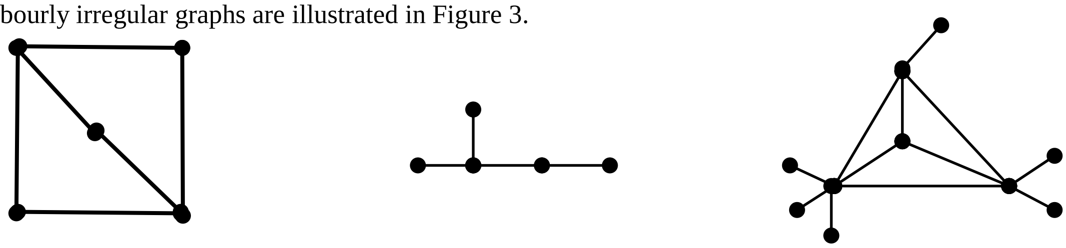


Figure 2

Yousef Alavi et. al [1], introduced this concept and studied some properties on highly irregular graphs in [1]. Some results on highly irregular bipartite graphs have also been established in [10]. One can easily note that any highly irregular graph G contains P_4 as an induced subgraph such that the pendant vertices of P_4 are the pendant vertices of G and the non-pendant vertices of P_4 are the vertices of maximum degree in G.

A connected graph G is said to be a *totally segregated graph*[9], if no two adjacent vertices of G have the same degree. Later, it has been independently studied by Gnaana Bhragsam and Ayyaswamy in [8]. They called this graph as a *neighbourly irregular graph* (or simply an *NI graph*).

Some neighbourly irregular graphs are illustrated in Figure 3.

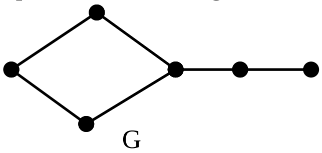


G_1 G_2 G_3

Figure 3

ISSN: 2456 – 3080
International Journal of Applied and Advanced Scientific Research
Impact Factor 5.255, Special Issue, February - 2017
International Conference on Advances in Theoretical and Applied Mathematics – ICATAM 2017
On 14th February 2017 Organized By
Madurai Sivakasi Nadars Pioneer Meenakshi Women's College, Poovanthi, Tamilnadu

A connected graph is said to be *support neighbourly irregular* (or simply *SNI*)[4], if no two vertices having same support are adjacent. A graph proving the existence of SNI graphs is shown in Figure 4.



G
Figure 4

A graph G is said to be a *balanced graph*[6], if every vertex in G has the same support. It is easy to observe that the complete bipartite graphs $K_{m,n}$ and any regular graphs are balanced graphs. A graph G is said to be *highly unbalanced,* if distinct vertices of G have distinct supports. For example, a highly unbalanced graph is shown in Figure 5.

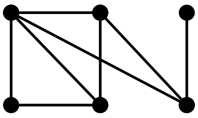


Figure 5

Balanced graphs are nothing but support – regular graphs with no restriction on the degrees of their vertices. Regular graphs are the only graphs with vertices of same degree and same support. In fact these two parameters are not dependent on each other. Two vertices of same degree need not have same support in a graph.

Even a pendant vertex in a graph may have same or more or less support compared to that of a vertex of degree two in the same graph. More surprisingly, in $K_{1,n}$, the pendant vertex as well as the center vertex have the same support n. What happens if the support of a vertex varies proportionately with its degree in a graph?

A graph G is said to be a *k – perfect degree support graph* (or simply a *k – pds graph*)[5], if for any vertex v in G, $\frac{s(v)}{d(v)} = k$. For example, the graph $C_4 \circ K_2$ shown in Figure 6 is a 3 – pds graph. In general, $C_n \circ K_2$ is a 3 – pds graph for any $n \geq 3$.

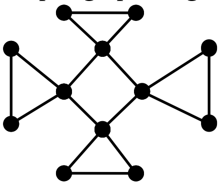


Figure 6: $C_4 \circ K_2$

A graph G is said to be a *(k,c) – linear degree support graph* (or simply a *(k,c) – lds graph*)[5], if for every vertex u in G, s(u) = k d(u) + c, for a fixed integer c. Or equivalently, G is (k,c) – lds if and only if for any two vertices u and v in G with $d(u) \neq d(v)$ in G, $\frac{s(u)-s(v)}{d(u)-d(v)} = k$. For example, the graph shown in Figure 7 is a (3, -1) – lds graph.

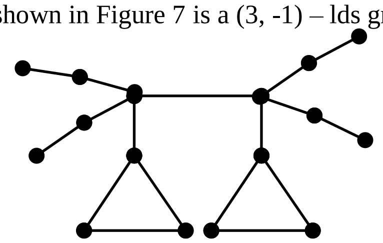


Figure 7

Note that any k – pds graph is a (k,0) – lds graph. In [5], it has been proved that a graph G is 2 – pds if and only if G is isomorphic to the cycle C_n, $n \geq 3$, or the subdivision graph $S_1(K_{1,3})$. In this paper, we study about k – pds irregular graphs.

2. Irregularity of k – pds Graphs:

Theorem 2.1:

K_2 is the only highly irregular k – pds graph.

Proof:

Suppose there exists a HI graph G which is k – pds for some $k \geq 1$. Then by the definition of a HI graph, G contains at least two pendant vertices. Also G contains P_4 as an induced subgraph with each of these pendant vertices is adjacent to a vertex of maximum degree Δ in G. Therefore, support of each of such pendant vertex is Δ which fixes the value of k as Δ in G.

Now consider the vertex(say v) of maximum degree Δ in G. The neighbours of v are of degrees 1,2,3,...,Δ in a highly irregular graph. Then support of v = 1+2+…+Δ = Δ(Δ+1)/2. But since G is a Δ – pds graph, the support of a maximum degree vertex must be Δ^2. This forces that Δ = 1. Since G is connected, we get $G \cong K_2$. Thus K_2 is the only highly irregular k – pds graph.

Theorem 2.2:

For any $k \geq 2$, there exists a NI k – pds graph.

Proof:

ISSN: 2456 – 3080
International Journal of Applied and Advanced Scientific Research
Impact Factor 5.255, Special Issue, February - 2017
International Conference on Advances in Theoretical and Applied Mathematics – ICATAM 2017
On 14th February 2017 Organized By
Madurai Sivakasi Nadars Pioneer Meenakshi Women's College, Poovanthi, Tamilnadu

Let k = 2. Since a graph is 2 – pds if and only it is C_n, $n \geq 3$, or $S_1(K_{1,3})$, we have the subdivision graph $S_1(K_{1,3})$ is the only NI 2- pds graph. For k > 2, construct a graph G_k with vertex set $V(G_k) = \{w, u_i, v_{ij}, w_m / 1 \leq i \leq k(k-1), 1 \leq j \leq k-2, 1 \leq m \leq k(k-1)/2\}$ and edge set $E(G_k) = \{wu_i, 1 \leq i \leq k(k-1)\} \cup \{u_iv_{ij}, 1 \leq i \leq k(k-1), 1 \leq j \leq k-2\} \cup \{w_mu_{2m-1}, w_mu_{2m}, 1 \leq m \leq k(k-1)/2\}$. In the constructed graph G_k, $d(w) = k(k-1)$ and $s(w) = k^2(k-1)$, $d(u_i) = k$ and $s(u_i) = k^2$, $d(w_m) = 2$ and $s(w_m) = 2$ and the pendant vertices v_{ij} have support k. Hence G_k is a k – pds graph. Also no two vertices of same degree are adjacent in G_k. Therefore G_k is a NI k – pds graph.

As an illustration, for the case k = 3, the 3 – pds graph G_3 is shown in Figure 8.

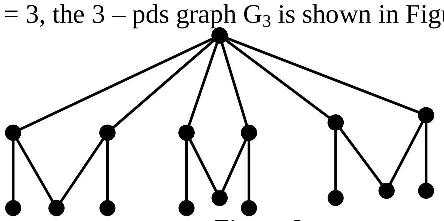


Figure 8

Theorem 2.3:

A k – pds graph is SNI if and only if it is an NI graph.

Proof:

Let G be any k – pds graph. Then s(u) = kd(u) for every vertex u in G.

Now, G is SNI ⇔ s(u) ≠ s(v), for any two adjacent vertices u and v in G.

⇔ kd(u) ≠ kd(v), for any two adjacent vertices u and v in G.

⇔ d(u) ≠ d(v), for any two adjacent vertices u and v in G.

⇔ G is NI.

Corollary 2.4:

For any $k \geq 2$, there exists a SNI k – pds graph.

Proof:

The result follows from Theorem 2.2 and Theorem 2.3.

Theorem 2.5:

There exists no highly unbalanced k – pds graph.

Proof:

Let G be a highly unbalanced k – pds graph. Then for any two vertices u and v in G, s(u) ≠ s(v). Since G is k – pds, for any vertex u in G, s(u) = k d(u). Therefore s(u) ≠ s(v) implies that d(u) ≠ d(v) for every two vertices u and v in G. But we know that, any graph contains at least two vertices of same degree. Hence there does not exist any highly unbalanced k – pds graph.

Theorem 2.6:

An RA graph is k – pds if and only if it is regular.

Proof:

Let G be an RA graph. Then by Result A, G is regular or biregular bipartite with each bipartition having vertices of same degree. It is easy to note that any biregular bipartite with each bipartition having vertices of same degree is a balanced graph. It is obvious that balanced graph which is not regular cannot be k – pds for any constant k. Therefore G is k – pds implies that G is regular. And the converse is obvious.

We have seen that a biregular bipartite with each bipartition having vertices of same degree cannot be a k – pds graph. But there do exist biregular k – pds graphs, an example of which is given in Figure 6. Our next theorem aims at characterizing an (a,b) – biregular graph which is k – pds too.

It is easy to note that in a k – pds biregular graph, any two vertices of degree a have the same number of neighbours of degree b and vice versa. Let u be any vertex of degree a in G. Let x denote the number of neighbours of u which are of degree a. That is, $x = |\{w \in N(u) / d(w) = a\}|$. Similarly let v be any vertex of degree b in G. Let y denote the number of neighbours of v which are of degree b. That is, $y = |\{w \in N(v) / d(w) = b\}|$.

Theorem 2.7:

An (a,b) – biregular graph is k – pds if and only if $\frac{x}{a} + \frac{y}{b} = 1$.

Proof:

Let G be any (a,b) – biregular k – pds graph. Let u and v be any two vertices in G of degree a and b respectively. Then $s(u) = ax+b(a-x) = ka$. Similarly, $s(v) = by+a(b-y) = kb$. Hence $\frac{ax+(a-x)b}{a} = \frac{by+a(b-y)}{b}$. This forces that $\frac{x}{a} + \frac{y}{b} = 1$. Conversely let G be an (a,b) – biregular graph with $\frac{x}{a} + \frac{y}{b} = 1$. Retracing the above steps, we get G is k – pds.

Theorem 2.8:

A graph G is k_1 – pds and (k_2, c) – lds for some constants k_1, k_2 and c if and only if G is regular.

Proof:

ISSN: 2456 – 3080
International Journal of Applied and Advanced Scientific Research
Impact Factor 5.255, Special Issue, February - 2017
International Conference on Advances in Theoretical and Applied Mathematics – ICATAM 2017
On 14th February 2017 Organized By
Madurai Sivakasi Nadars Pioneer Meenakshi Women's College, Poovanthi, Tamilnadu

Let G be any graph. Suppose G is a k_1 – pds and (k_2, c) – lds for some constants k_1, k_2 and c. If possible let G be not regular. Then there exist at least two vertices u and v in G such that d(u) ≠ d(v). Since G is and (k_2, c) – lds, s(u) = k_1d(u) = k_2d(u) + c, which implies, d(u) = $\frac{c}{k_1-k_2}$, which is a constant. In a similar manner, we get d(v) = $\frac{c}{k_1-k_2}$. Therefore we get d(u) = d(v), which leads to a contradiction. Hence G is regular. Conversely, if G is r – regular, then it is r – pds and (r,0) – lds graph.

Theorem 2.9:

A graph G is (k_1, c_1) – lds and (k_2, c_2) – lds for some constants k_1, k_2, c_1 and c_2 such that $k_1 \neq k_2$ and $c_1 \neq c_2$ if and only if G is regular.

Proof:

Suppose a graph G is (k_1, c_1) – lds and (k_2, c_2) – lds for some constants k_1, k_2, c_1 and c_2 such that $k_1 \neq k_2$ and $c_1 \neq c_2$. Then for any vertex u in G, s(u) = k_1d(u) + c_1 = k_2d(u) + c_2, which implies d(u) = $\frac{c_2-c_1}{k_1-k_2}$, which is always a constant. Therefore G is regular. Conversely, any r – regular graph is (r,0) – lds and (1, r(r-1)) – lds.

It is easy to note that any r – regular graph is (k, r(r – k)) – lds, for any k, $1 \leq k \leq r$.

Corollary 2.10:

A graph G is (k_1, c) – lds and (k_2, c) – lds for some constants k_1 and k_2 such that $k_1 \neq k_2$ if and only if G is trivial.

References:

1. Yousef Alavi, Gary Chartand, F. R. K. Chung, Paul Erdos, R. L.Graham and O. R. Oellermann, Highly irregular graphs, Journal of Graph Theory 11(1987), 235-249.
2. Selvam Avadayappan and M. Bhuvaneshwari, Some results on degree splitting graphs, International Journal of Advanced and Innovative Research, Vol. 5, Issue 3, March 2016.
3. Selvam Avadayappan and M. Bhuvaneshwari, Perfect degree support product graphs, International Journal for Research in Applied Science and Engineering Technology, vol. 4, Issue 3, March 2016.
4. Selvam Avadayappan and M. Bhuvaneshwari, Support Neighbourly irregular graphs, International Journal of Scientific Research and Management, Volume 4, Issue 3, March 2016.
5. Selvam Avadayappan and M. Bhuvaneshwari, Perfect degree support graphs, (Preprint).
6. Selvam Avadayappan and G. Mahadevan, Highly unbalanced graphs, ANJAC Journal of Sciences, 2(1), 23 – 27, 2003.
7. R. Balakrishnan and K. Ranganathan, A Text Book of graph Theory, Springer-Verlag, New York, Inc (1999).
8. S.Gnaana Bhragsam and S.K. Ayyaswamy, Neighbourly irregular graphs, Indian Journal of Pure and Applied Mathematics, 35(3): 389 – 399, March 2004.
9. D. E. Jackson and Roger Entringer, Totally segregated graphs, Congressus Numerantium 55(1986), pp. 159 – 165.
10. A. Selvam, Highly irregular bipartite graphs, Indian Journal of Pure and Applied Mathematics, 27(6): 527 – 536, June 1996.