ISSN: 2456 – 3080
International Journal of Applied and Advanced Scientific Research
Impact Factor 5.255, Special Issue, February - 2017
International Conference on Advances in Theoretical and Applied Mathematics – ICATAM 2017
On 14th February 2017 Organized By
Madurai Sivakasi Nadars Pioneer Meenakshi Women's College, Poovanthi, Tamilnadu

ANALYTICAL EXPRESSIONS OF A NON-LINEAR BOUNDARY VALUE PROBLEM FOR REACTIVE THIRD GRADE FLUID FLOW AND HOMOTOPY PERTURBATION METHOD

K. M. Dharmalingam* & M. Subha**
* Department of Mathematics, The Madura College, Madurai, Tamilnadu
** Department of Mathematics, MSNPM Women's College, Poovanthi, Tamilnadu

Cite This Article: K. M. Dharmalingam & M. Subha, "Analytical Expressions of a Non-Linear Boundary Value Problem for Reactive Third Grade Fluid Flow and Homotopy Perturbation Method", International Journal of Applied and Advanced Scientific Research, Special Issue, February, Page Number 4-15, 2017.

Abstract:

In this article we present a mathematical analysis of the steady flow of a reactive third-grade liquid between two parallel isothermal plates. It is assumed that the reaction is exothermic under Arrhenius kinetics, neglecting the consumption of the material. The governing simultaneous non-linear differential equations are solved using the Homotopy perturbation method (HPM). The approximate analytical expressions of the dimensionless velocity and the dimensionless temperature fields are also derived by using the HPM. The effects of various non-Newtonian fluid parameters on both the dimensionless velocity and dimensionless temperature fields are constructed and discussed. Besides, skin-friction parameter and the Nusselt number are discussed analytically and graphically. Our analytical results show satisfactory agreement when compared with established work.

Key Words: Channel Flow, Third Grade Fluid, Arrhenius Kinetics, Nusselt Number & Homotopy Perturbation Method

1. Introduction:

Many fluids show non-Newtonian behavior, and so their rheological properties are not well modeled by Navier–Stokes equations in engineering and industrial applications [1-3]. The past few decades have witnessed significant research on flows of non-Newtonian fluids for two principal reasons: (i) their non-linearity in the inertial part, and (ii) non-linearity in the surface forces of the governing equations. Two crucial factors in handling non-linear forces are thermal criticality and heat transfer [5, 9]. Rajagopal [11] studied in substantial detail the general thermodynamics and stability of fluids of the differential type. His studies had fluids of third grade as special cases. In [5], Fosdick and Rajagopal give a complete thermodynamic analysis of the constitutive function for problems in heat transfer for fluids of third grade. Reports of similar studies for non-Newtonian fluids can be found in [11, 15]. Mathematically, as regards non-Newtonian fluids, the thermal boundary layer equation is a nonlinear problem. Also, thermal criticality is a complex physical process. Insight into this process can be gained by studying the long-term behavior of the solutions in space. The present work considers steady flow of a reactive third grade liquid between two parallel isothermal plates. The objective of this work is to derive the analytical expressions of such a flow using the Homotopy perturbation method. The graphical and the analytical expressions of skin-friction parameter and the Nusselt number are also discussed.

2. Mathematical Formulation of the Problem:

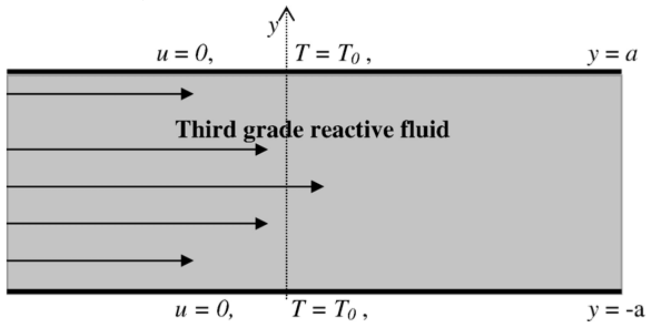


Figure 1: Geometry of the Problem

We consider the steady flow of an incompressible third grade reactive fluid between two parallel isothermal plates (see Fig. 1). The fluid motion is assumed induced by applied axial pressure gradient. The reference x-axis is parallel to the plate and y-axis normal to it. In the case of hydrodynamically and thermally developed flow, the velocity and temperature fields depend on y only. Following ([4-5, 7, 9]) and neglecting the reacting viscous fluid consumption, the one-dimensional governing equations for the momentum and heat balance are:

$$\mu \frac{d^2u}{dy^2} + 6\beta_3 \frac{d^2u}{dy^2}\left(\frac{du}{dy}\right)^2 = \frac{dP}{dx} \qquad (1)$$

$$k\frac{d^2T}{dy^2} + \left(\frac{du}{dy}\right)^2\left(\mu + 2\beta_3\left(\frac{du}{dy}\right)^2\right) + QC_0 A e^{-\frac{E}{RT}} = 0 \qquad (2)$$

Subject to the boundary conditions:

$$\frac{du}{dy}(0)=\frac{dT}{dy}(0)=0, u(a)=0, T(a)=T_0, \tag{3}$$

where the additional Arrhenius kinetics term in energy balance equation is due to [6,13]. Here T is the absolute temperature, U the fluid characteristic velocity, T_0 the plate temperature, k the thermal conductivity of the material, Q the heat of reaction, A the rate constant, E the activation energy, R the universal gas constant, C_0 the initial concentration of the reactant species, a the channel half width, β_3 the material coefficient, P the modified pressure and μ is the fluid dynamic viscosity coefficient. We introduce the dimensionless variables into the eqns. (1) - (3) are as follows:

$$\theta=\frac{E(T-T_0)}{RT_0^2}, \bar{y}=\frac{y}{a}, \lambda=\frac{QEAa^2C_0e^{-\frac{E}{RT_0}}}{T_0^2Rk}, W=\frac{u}{UG} \tag{4}$$

$$m=\frac{\mu G^2U^2e^{\frac{E}{RT_0}}}{QAa^2C_0}, \varepsilon=\frac{RT_0}{E}, G=-\frac{a^2}{\mu U}\frac{dP}{dx}, \gamma=\frac{\beta_3U^2G^2}{a^2\mu} \tag{5}$$

The dimensionless governing equation together with the corresponding boundary conditions (neglecting the bar symbol for clarity) are:

$$\frac{d^2W}{dy^2}+6\gamma\frac{d^2W}{dy^2}\left(\frac{dW}{dy}\right)^2=-1 \tag{6}$$

$$\frac{d^2\theta}{dy^2}+\lambda\left[e^{\left(\frac{\theta}{1+\varepsilon\theta}\right)}+m\left(\frac{dW}{dy}\right)^2\left(1+2\gamma\left(\frac{dW}{dy}\right)^2\right)\right]=0 \tag{7}$$

With

$$\frac{dW}{dy}(0)=\frac{d\theta}{dy}(0)=0, W(1)=0, \theta(1)=0 \tag{8}$$

Where λ, ε, γ, m represent the Frank–Kamenetskii parameter, activation energy parameter, the dimensionless non-Newtonian parameter and the viscous heating parameter, respectively. The physical quantities of interest in this problem are the skin-friction parameter (C_f) and the Nusselt number (Nu) which are defined as follows [1]:

$$C_f=\frac{al_w}{\mu UG}=-W'(1) \tag{9}$$

$$Nu=\frac{aE_{qw}}{kRT_o^2}=-\theta'(1) \tag{10}$$

3. Approximate Analytical Expressions of the Non-Linear Boundary Value Problem Using Homotopy Perturbation Method:

Linear and non-linear phenomena are of fundamental importance in various fields of science and engineering. Most models of real-life problems are still very difficult to solve. Therefore, approximate analytical solutions such as Homotopy perturbation method (HPM) [16-23] were introduced. HPM is the most effective and convenient ones for both linear and non-linear equations. This method is based on assuming a small parameter. The majority of non-linear problems, especially those having strong non-linearity, have no small parameters at all and the approximate solutions obtained by the perturbation methods, in most cases, are valid only for small values of the small parameter. Generally, the perturbation solutions are uniformly valid as long as a scientific system parameter is small. However, we cannot rely fully on the approximations, because there is no criterion for the existence of a small parameter. Thus, it is essential to check the validity of the approximations numerically and/or experimentally. HPM was proposed to overcome these difficulties.

Recently, many authors have applied HPM to solve the non-linear boundary value problem in physics and engineering sciences [16-19]. HPM is also used to solve some of the non-linear problem in physical sciences [16-18]. This method is a combination of Homotopy in topology and classic perturbation techniques. Ji-Huan He used HPM to solve the Lighthill equation [20], the Diffusion equation [21] and the Blasius equation [22]. HPM is unique in its applicability, accuracy and efficiency. Homotopy perturbation method uses the imbedding parameter p as a small parameter, and only a few iterations are needed to search for an asymptotic solution. The approximate analytical expressions of the eqns. (6)- (8) using the Homotopy perturbation method is as follows [23-26]:

$$W(y)=\frac{1}{2}(1-y^2)(1+\gamma+\gamma y^2) \tag{11}$$

ISSN: 2456 – 3080
International Journal of Applied and Advanced Scientific Research
Impact Factor 5.255, Special Issue, February - 2017
International Conference on Advances in Theoretical and Applied Mathematics – ICATAM 2017
On 14th February 2017 Organized By
Madurai Sivakasi Nadars Pioneer Meenakshi Women's College, Poovanthi, Tamilnadu

$$\theta(y)=(1-y^2)\left(\frac{\lambda}{2}\right)+\left[-\frac{\lambda^2}{2}+\frac{\lambda^3\in}{4}\right]\left(\frac{y^2}{2}\right)+\left[\frac{\lambda^2}{2}-\frac{\lambda^3\in}{2}-\lambda m\right]\left(\frac{y^4}{12}\right)$$
$$+\left[\frac{\lambda^3\in}{4}-4\lambda m\gamma-2\lambda\gamma m\right]\left(\frac{y^6}{30}\right)+\left[-4\lambda m\gamma^2-16\gamma^2\lambda m\right]\left(\frac{y^8}{56}\right) \quad (12)$$
$$-\left(\frac{48}{90}\right)(\lambda m\gamma^3y^{10})-\left(\frac{64}{132}\right)(\lambda m\gamma^4y^{12})-\left(\frac{32}{182}\right)(\lambda\gamma^5y^{14})+B$$

Where

$$B=\left[\frac{\lambda^2}{2}-\frac{\lambda^3\in}{4}\right]\left(\frac{1}{2}\right)+\left[-\frac{\lambda^2}{2}+\frac{\lambda^3\in}{2}+\lambda m\right]\left(\frac{1}{12}\right)+\left[-\frac{\lambda^3\in}{4}+4\lambda m\gamma+2\lambda\gamma m\right]\left(\frac{1}{30}\right) \quad (13)$$
$$+\left[4\lambda m\gamma^2+16\gamma^2\lambda m\right]\left(\frac{1}{56}\right)+\left(\frac{48}{90}\right)(\lambda m\gamma^3)+\left(\frac{64}{132}\right)(\lambda m\gamma^4)+\left(\frac{32}{182}\right)(\lambda\gamma^5)$$

The analytical expressions of the skin-friction parameter and the Nusselt number respectively using the eqns. (9) and (10) are as follows:

$$C_f=(1+2\gamma) \quad (14)$$

$$Nu=\lambda+\frac{\lambda^2}{3}-\frac{2\lambda^3\in}{15}+\frac{\lambda m}{3}+\frac{6\lambda m\gamma}{5}+\frac{20\lambda m\gamma^2}{7}+\frac{16\lambda m\gamma^3}{3}+\frac{64\lambda m\gamma^4}{11}+\frac{32\lambda\gamma^5}{13} \quad (15)$$

4. Previous Work:

The approximate analytical solutions for the dimensionless velocity and dimensionless temperature fields using the Hermite-Pade' approximations are as follows ([1, 7-10, 13-14]):

$$W(y)=\frac{1}{2}-\frac{1}{2}y^2+\left[\frac{1}{2}(y^4-1)\right]\gamma-\left[2(y^2-1)(y^2+y+1)(y^2-y+1)\right]\gamma^2 \quad (16)$$
$$+\left[12(y^2-1)(y^2+1)(y^4+1)\right]\gamma-\left[\begin{matrix}88(y^2-1)(y^4+y^3+y^2+y+1)\\(y^4-y^3+y^2-y+1)\end{matrix}\right]\gamma^4+O(\gamma^5)$$

$$\theta(y)=\left(\begin{matrix}\frac{16}{91}m\gamma^5y^{14}+\frac{16}{33}m\gamma^4y^{12}-\frac{8}{15}m\gamma^3y^{10}+\frac{3}{14}m\gamma^2y^8+\frac{1}{15}m\gamma y^6-\frac{1}{12}my^4\\-\frac{1}{2}y^2+\frac{16}{91}m\gamma^5-\frac{16}{33}m\gamma^4+\frac{8}{15}m\gamma^3-\frac{3}{14}m\gamma^2-\frac{1}{15}m\gamma+\frac{1}{12}m+\frac{1}{2}\end{matrix}\right)\lambda$$
$$+\left(\begin{matrix}\frac{1}{1365}m\gamma^5y^{16}-\frac{8}{3003}m\gamma^4y^{14}+\frac{2}{495}m\gamma^3y^{12}-\frac{1}{420}m\gamma^2y^{10}-\frac{1}{840}m\gamma y^8\\+\frac{1}{360}my^6+\frac{1}{24}y^4-\frac{8}{91}y^2m\gamma^5+\frac{8}{33}y^2m\gamma^4-\frac{4}{15}y^2m\gamma^3+\frac{3}{28}y^2m\gamma^2\\+\frac{1}{30}y^2m\gamma-\frac{1}{24}y^2m-\frac{1}{4}y^2+\frac{17}{195}m\gamma^5-\frac{240}{1001}m\gamma^4+\frac{26}{99}m\gamma^3-\frac{11}{105}m\gamma^2\\-\frac{9}{280}m\gamma+\frac{7}{180}m+\frac{5}{24}\end{matrix}\right)\lambda^2+O(\lambda^3) \quad (17)$$

5. Results and Discussion:

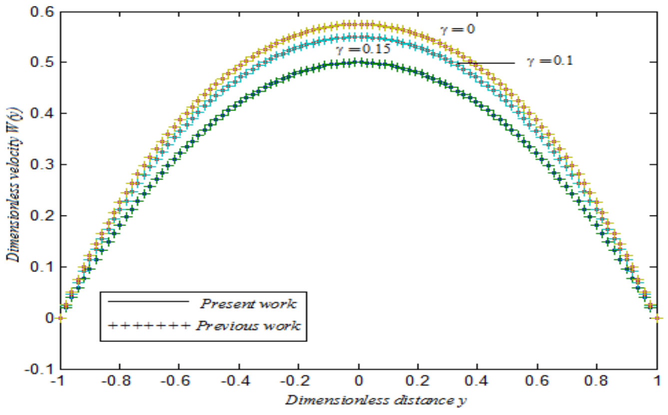


Figure 2: Dimensionless velocity $W(y)$ versus the dimensionless distance y . The velocity profiles are computed using the eqn. (11) for various values of the dimensionless non -Newtonian parameter γ.

ISSN: 2456 – 3080
International Journal of Applied and Advanced Scientific Research
Impact Factor 5.255, Special Issue, February - 2017
International Conference on Advances in Theoretical and Applied Mathematics – ICATAM 2017
On 14th February 2017 Organized By
Madurai Sivakasi Nadars Pioneer Meenakshi Women's College, Poovanthi, Tamilnadu

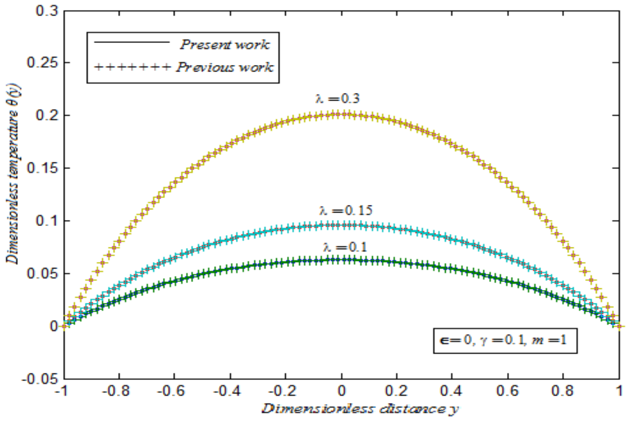
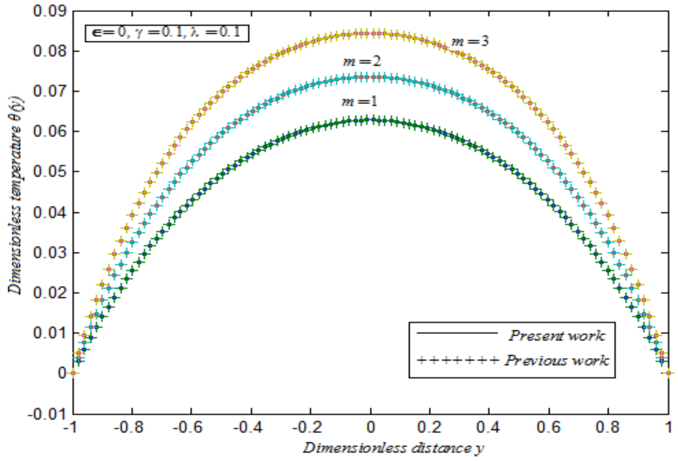
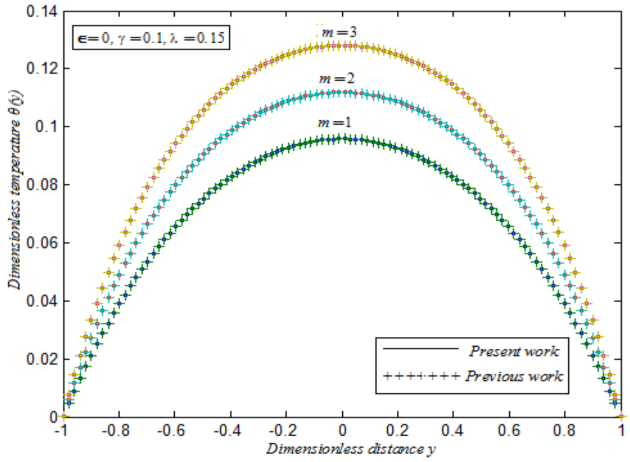


Figure 3: Dimensionless temperature $\theta(y)$ versus the dimensionless distance y. The temperature profiles are computed by using the eqns. (11) and (12) for various values of the Frank- Kamenetskii parameter λ and in some fixed values of the other dimensionless parameters.

(a)



(b)



(c)

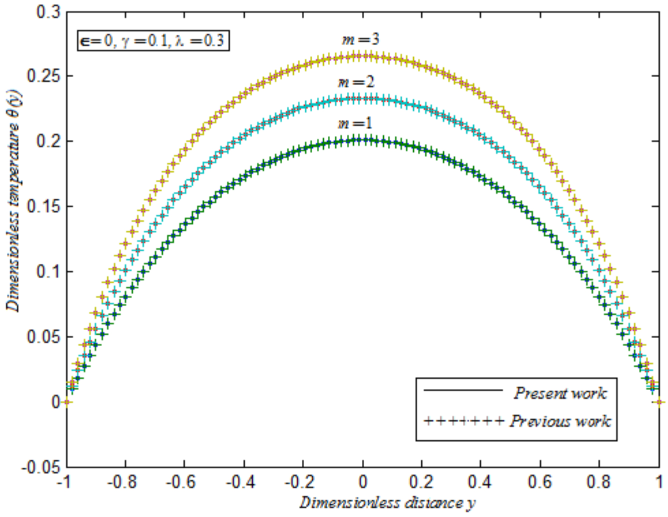
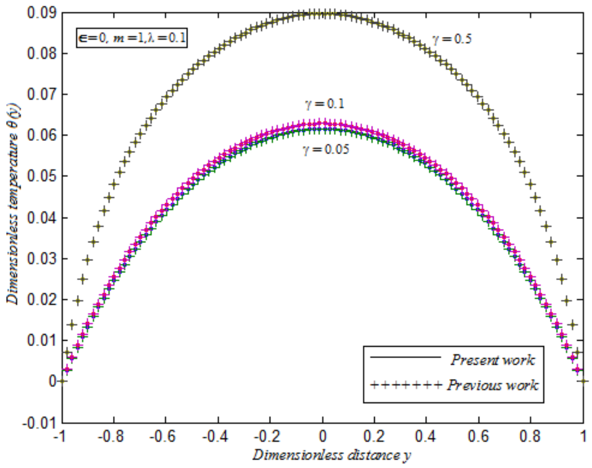


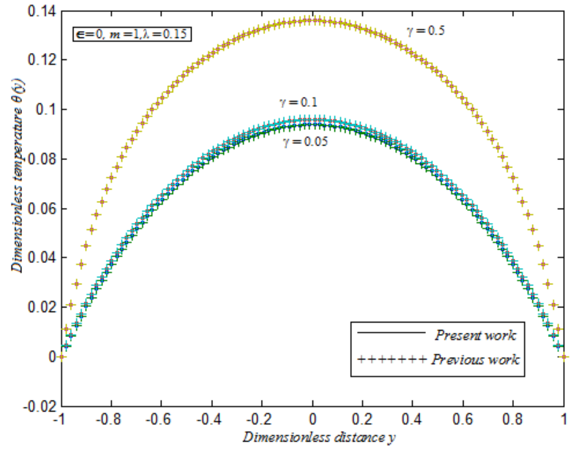
Figure 4: Dimensionless Temperature $\theta(y)$ profile versus the dimensionless distance y. The temperature profiles are computed using the eqns. (11) and (12) for various values of the viscous heating parameter m and in some fixed values of the other dimensionless parameters, for various values of the Frank- Kamenetskii parameter λ :(a) $\lambda = 0.1$ (b) $\lambda = 0.15$ and (c) $\lambda = 0.3$.

ISSN: 2456 – 3080
International Journal of Applied and Advanced Scientific Research
Impact Factor 5.255, Special Issue, February - 2017
International Conference on Advances in Theoretical and Applied Mathematics – ICATAM 2017
On 14th February 2017 Organized By
Madurai Sivakasi Nadars Pioneer Meenakshi Women's College, Poovanthi, Tamilnadu

(a)



(b)



(c)

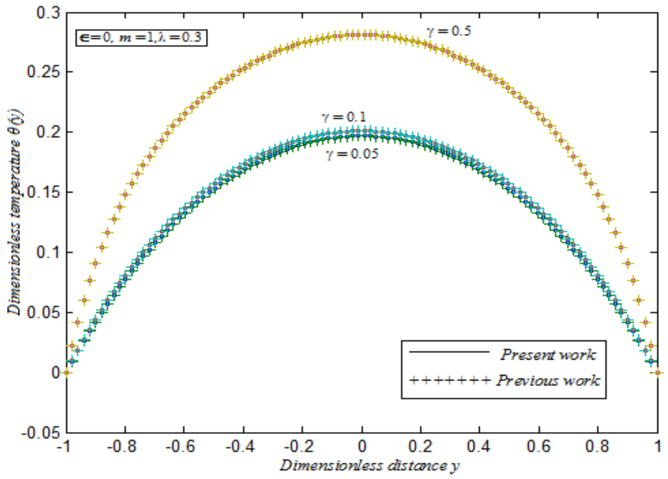


Figure 5: Dimensionless Temperature $\theta(y)$ profile versus the dimensionless distance y. The temperature profiles are computed using the eqns. (11) and (12) for various values of the dimensionless non- Newtonian parameter γ and in some fixed values of the other dimensionless parameters, and various values of the Frank- Kamenetskii parameter λ: (a) $\lambda = 0.1$ (b) $\lambda = 0.15$ and (c) $\lambda = 0.3$.

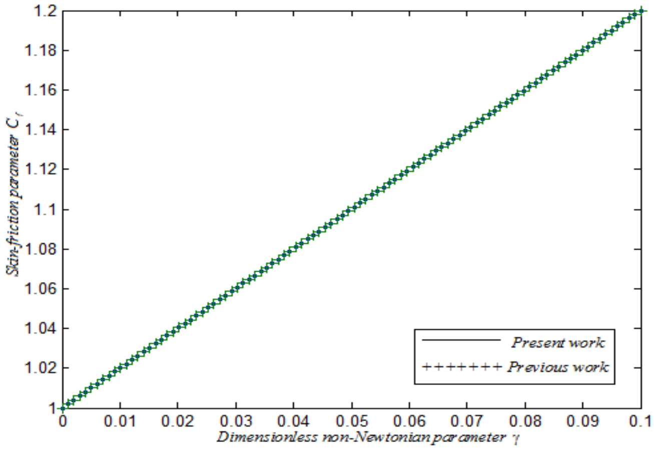


Figure 6: Variation of the dimensionless skin-friction parameter C_f with respect to the dimensionless non-Newtonian parameter γ using the eqn (14).

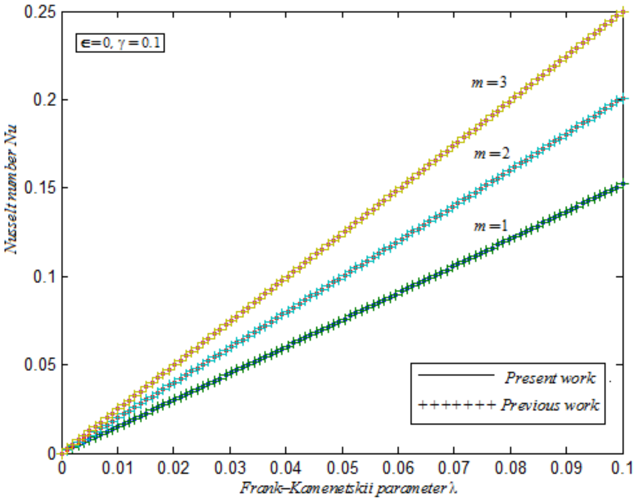


Figure 7: Variation of the Nusselt number Nu versus the dimensionless Frank- Kamenetskii parameter λ. The curves are plotted using the eqn.(15) for various values of the dimensionless viscous heating parameter m and in some fixed values of the other dimensionless parameters.

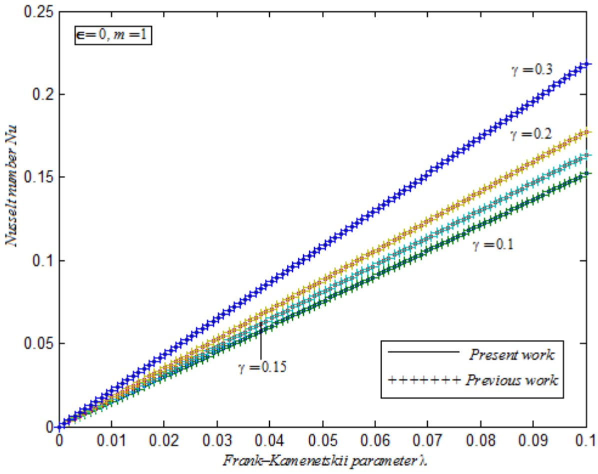


Figure 8: Variation of the Nusselt number Nu versus the dimensionless Frank- Kamenetskii parameter λ. The curves are plotted using the eqn.(15) for various values of the dimensionless non- Newtonian parameter γ and in some fixed values of the other dimensionless parameters.

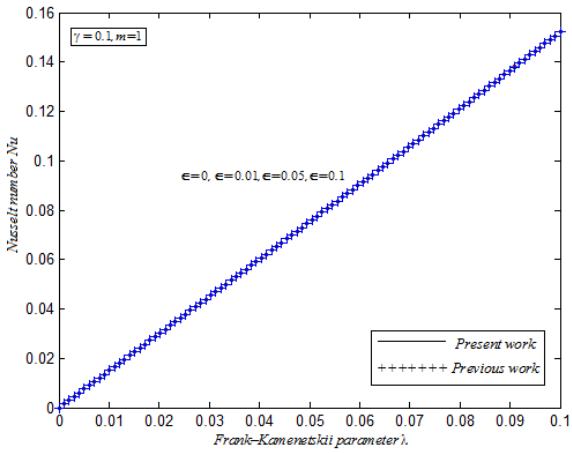


Figure 9: Variation of the Nusselt number Nu versus the dimensionless Frank- Kamenetskii parameter λ. The curves are plotted using the eqn.(15) for various values of the dimensionless activation energy parameter $\in$ and in some fixed values of the other dimensionless parameters.

ISSN: 2456 – 3080
International Journal of Applied and Advanced Scientific Research
Impact Factor 5.255, Special Issue, February - 2017
International Conference on Advances in Theoretical and Applied Mathematics – ICATAM 2017
On 14th February 2017 Organized By
Madurai Sivakasi Nadars Pioneer Meenakshi Women's College, Poovanthi, Tamilnadu

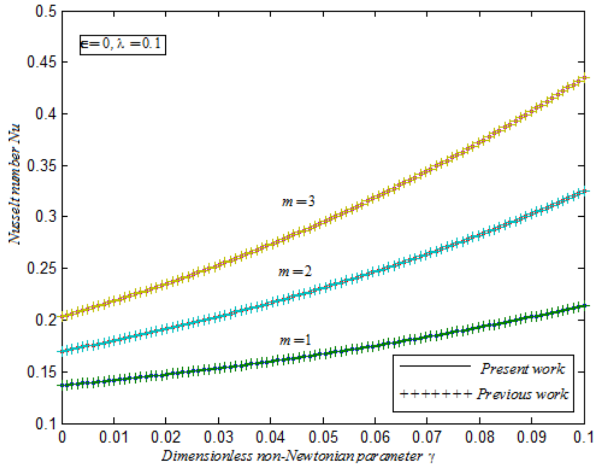


Figure 10: Variation of the Nusselt number Nu versus the dimensionless non-Newtonian parameter γ. The curves are plotted using the eqn.(15) for various values of the dimensionless viscous heating parameter m and in some fixed values of the other dimensionless parameters.

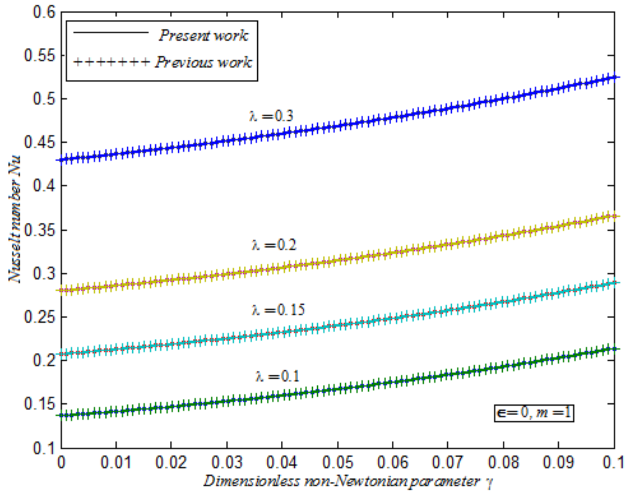


Figure 11: Variation of the Nusselt number Nu versus the dimensionless non-Newtonian parameter γ. The curves are plotted using the eqn.(15) for various values of the dimensionless Frank- Kamenetskii parameter λ and in some fixed values of the other dimensionless parameters.

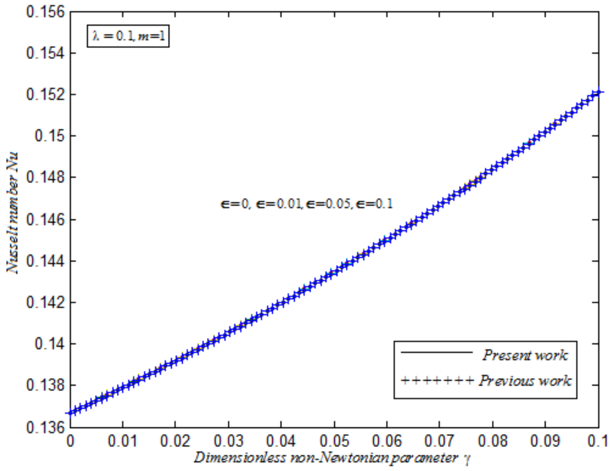


Figure 12: Variation of the Nusselt number Nu versus the dimensionless non-Newtonian parameter γ. The curves are plotted using the eqn.(15) for various values of the dimensionless activation energy parameter $\in$ and in some fixed values of the other dimensionless parameters.

Figure 1 represents the steady flow of an incompressible third grade reactive fluid placed between two parallel isothermal plates. Fig. 2 shows the dimensionless velocity $W(y)$ versus the dimensionless distance y. From Fig.1, it is evident that, when the dimensionless non- Newtonian parameter γ increases the corresponding dimensionless velocity $W(y)$ decreases. Figure 3 through figure 5 show the dimensionless temperature $\theta(y)$ versus the dimensionless distance y. From figure 3 it is noted that when the dimensionless Frank- Kamenetskii parameter λ increases, the corresponding temperature also increases in some fixed values of the other dimensionless parameters like activation energy parameter $\in$, non- Newtonian parameter γ and the viscous heating parameter m. From figures 4(a) through 4(c), it is observed that when the viscous heating parameter m increases the dimensionless temperatures $\theta(y)$ also increases in some fixed values of the other dimensionless parameters. From figures 5(a) through 5(c), it follows that when the dimensionless non-Newtonian parameter γ increases, the temperature profiles $\theta(y)$ also increases in some fixed values of the other dimensionless parameters. Fig. 6 is the variation of the dimensionless skin-friction parameter C_f with respect to the dimensionless non-Newtonian parameter γ. Figure 7 through figure 10 indicate the Nusselt number Nu versus the dimensionless Frank- Kamenetskii parameter λ. From figure7, it is understand that when dimensionless viscous heating parameter m increases the Nusselt number Nu also increases in some fixed values of the other dimensionless parameters. From figure 8, it is evident that when the dimensionless non-Newtonian parameter γ increases the corresponding Nusselt number Nu also increases in some fixed values of the other dimensionless parameters. From figure 9, it is observed that when the dimensionless activation energy parameter $\in$ increases, the Nusselt number Nu also increases and it coincides for all values of $\in$ in some fixed values of the other dimensionless parameters. Figure 10 through figure 12 demonstrates that the Nusselt number Nu versus the dimensionless non-Newtonian parameter γ. From figure 10, it is inferred that when the dimensionless viscous heating parameter m increases the Nusselt number Nu also increases in some fixed values of the other dimensionless parameters. From figure 11, it is noticed that when the dimensionless Frank- Kamenetskii parameter λ increases the Nusselt number Nu also increases in some fixed values of the other dimensionless parameters. From figure 12, it is evident that when the dimensionless activation energy parameter $\in$ increases the Nusselt number Nu also increases and it coincides for all values of $\in$ in some fixed values of the other dimensionless parameters.

6. Conclusion:

The non-linear boundary value problem for the steady flow of a reactive third-grade liquid between parallel isothermal plates has been solved analytically and compared with the Hermite-Pade approximation. The approximate analytical expressions of the dimensionless velocity and dimensionless temperature profiles can be derived by using the Homotopy perturbation method. The graphical representation of the fluid flow is also discussed. Also the mathematical expressions of the dimensionless skin-frication parameter and the Nusselt numbers are derived analytically and graphically. A satisfactory agreement is noted between these two methods. The main contributions to knowledge in this paper are:

- ✓ An increase in the non-Newtonian material parameter decreases the dimensionless velocity within the flow channel; and
- ✓ An increase in the non-Newtonian material parameter increases the dimensionless temperature profiles within the flow channel.

The Homotopy perturbation method is an extremely simple as well as a promising method to solve other strongly non-linear boundary value problem in science and engineering.

7. Acknowledgement:

Researchers express their gratitude to the Secretary Shri. S. Natanagopal, Madura College Board, Madurai, Dr. J. Suresh, The Principal and Dr. S. Muthukumar, Head of the Department of Mathematics, The Madura College, Madurai, Tamil Nadu, India and Mrs. C. Ramalakshmi, The Principal, Mrs. K. Sornadevi, Head of the Department of Mathematics, Madurai Sivakasi Nadars Pioneer Meenakshi Women's College, Poovanthi, Sivagangai District, Tamil Nadu, India for their constant support and encouragement.

8. References:

1. O. D. Makinde, Hermite-Padé approximation approach to thermal criticality for a reactive third-grade liquid in a channel with isothermal walls, International Communications in Heat and Mass Transfer, 34 (2007): 870-877.
2. A.M. Siddiqui, M. Ahmed, Q.K. Ghori, Couette and Poiseuille flows for non Newtonian fluids, Int. J. Nonlinear Sci. Numer. Simul. 7 (1) (2006): 15–26.
3. A. M. Siddiqui, R. Mahmood, Q. K. Ghori, Thin film flow of a third grade fluid on a moving belt by He's Homotopy perturbation method, Int. J. Nonlinear Sci. Numer. Simul. 7(1) (2006): 1-8.
4. E. Kreyszig, E.J. Norminton, Advanced engineering mathematics, John Wiley & Sons, Inc., New York, 1994.
5. R.L. Fosdick, K.R. Rajagopal, Thermodynamics and stability of fluids of third grade, Proc. R. Soc. Lond., A 339 (1980): 351.
6. D.A. Frank Kamenetskii, Diffusion and heat transfer in chemical kinetics, Plenum Press, New York, 1969.

7. A.J. Guttamann, C. Domb, J.K. Lebowitz (Eds.), Asymptotic analysis of power series expansions-Phase transitions and critical phenomena, Academic Press, New York, 1989: 1–234.
8. O.D. Makinde, Exothermic explosions in a slab: a case study of series summation technique, Int. Commun. Heat Mass Transf. 31 (8) (2004): 1227–1231.
9. O.D. Makinde, Thermal ignition in a reactive viscous flow through a channel filled with a porous medium, ASME, J. Heat Transfer 128 (2006): 601–604.
10. O.D. Makinde, On steady flow of a reactive variable viscosity fluid in a cylindrical pipe with an isothermal wall, Int. J. Numer. Meth. Heat Fluid Flow 17 (2) (2007): 187–194.
11. M. Massoudi, I. Christe, Effects of variable viscosity and viscous dissipation on the flow of a third grade fluid in a pipe, Int. J. Non-Linear Mech. 30 (1995): 687.
12. K.R. Rajagopal, On boundary conditions for fluids of the differential type: Navier-Stokes equations and related non-linear problems, Plenum press, New York, 1995: 273.
13. A.V. Sergeyev, D.Z. Goodson, Summation of asymptotic expansions of multiple valued functions using algebraic approximations–application to anharmonic oscillators, J. Phys., A 31 (1998): 4301–4317.
14. Y. Tourigny, P.G. Drazin, The asymptotic behaviour of algebraic approximants, Proc. R. Soc. Lond., A 456 (2000): 1117–1137.
15. M. Yurusoy, M. Pakdemirli, Approximate analytical solutions for the flow of a third grade fluid in a pipe, Int. J. Non-Linear Mech. 37 (2002): 187.
16. Q. K. Ghori, M. Ahmed, and A. M. Siddiqui, Application of Homotopy perturbation method to squeezing flow of a Newtonian fluid, Int. J. Nonlinear Sci. Numer. Simulat, 8(2007): 179–184.
17. T. Ozis, and A. Yildirim, A Comparative study of He's Homotopy perturbation method for determining frequency-amplitude relation of a nonlinear oscillator with discontinuities, Int. J. Nonlinear Sci. Numer. Simulat, 8 (2007): 243–248.
18. S. J. Li, and Y. X. Liu, An Improved approach to nonlinear dynamical system identification using PID neural networks, Int. J. Nonlinear Sci. Numer. Simulat, 7 (2006): 177–182.
19. M. M. Mousa, S. F. Ragab, and Z. Nturforsch, Application of the Homotopy Perturbation method to linear and nonlinear Schrödinger equations, Zeitschriftfür Naturforschung, 63(2008): 140–144.
20. J. H. He, Homotopy perturbation technique, Comp Meth. Appl. Mech. Eng, 178(1999): 257–262.
21. J. H. He, Homotopy perturbation method: a new nonlinear analytical technique, Appl. Math. Comput, 135(2003): 73–79.
22. J. H. He., A simple perturbation approach to Blasius equation, Appl. Math. Comput, 140(2003): 217–222.
23. P. D. Ariel, Alternative approaches to construction of Homotopy perturbation Algorithms, Non-linear. Sci. Letts. A., 1(2010): 43–52.
24. V. Ananthaswamy, R. Thenmozhi and R. Seethalakshmi, Analytical expressions for combined effect of buoyancy force, International Journal of Modern Mathematical Sciences, 14(1) (2016): 77-99.
25. V. Ananthaswamy, T. Iswarya, Analytical expressions of mass transfer effects on unsteady flow past an accelerated vertical porous plate with suction, Nonlinear Studies, 23(1) (2016): 73-86.
26. V. Ananthaswamy, T. Iswarya, Analytical expressions of the effect of radiation on free convective flow of heat and mass transfer, Nonlinear Studies, 23(1) (2016): 133-147.

Appendix: A

Basic Concept of the Homotopy Perturbation Method:

To explain this method, let us consider the following function:

$$D_o(u) - f(r) = 0, \quad r \in \Omega \tag{A.1}$$

with the boundary conditions of

$$B_o\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma \tag{A.2}$$

Where D_o is a general differential operator, B_o is a boundary operator, $f(r)$ is a known analytical function and Γ is the boundary of the domain Ω. In general, the operator D_o can be divided into a linear part L and a non-linear part N.

The eqn. (A.1) can therefore be written as

$$L(u) + N(u) - f(r) = 0 \tag{A.3}$$

By the Homotopy technique, we construct a Homotopy $v(r,p): \Omega \times [0,1] \rightarrow \Re$ that satisfies

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[D_o(v) - f(r)] = 0 \tag{A.4}$$

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0 \tag{A.5}$$

where $p \in [0, 1]$ is an embedding parameter, and u_0 is an initial approximation of the eqn. (A.1) that satisfies the boundary conditions. From the eqns. (A.4) and (A.5), we have

ISSN: 2456 – 3080
International Journal of Applied and Advanced Scientific Research
Impact Factor 5.255, Special Issue, February - 2017
International Conference on Advances in Theoretical and Applied Mathematics – ICATAM 2017
On 14th February 2017 Organized By
Madurai Sivakasi Nadars Pioneer Meenakshi Women's College, Poovanthi, Tamilnadu

$$H(v,0)=L(v)-L(u_0)=0 \quad (A.6)$$

$$H(v,1)=D_o(v)-f(r)=0 \quad (A.7)$$

When p=0, eqn. (A.4) and eqn. (A.5) become linear equations. When p =1, they become non-linear equations. The process of changing p from zero to unity is that of $L(v)-L(u_0)=0$ to $D_o(v)-f(r)=0$. We first use the embedding parameter p as a small parameter and assume that the solutions of eqn. (A.4) and eqn. (A.5) can be written as a power series in p :

$$v=v_0+pv_1+p^2v_2+... \quad (A.8)$$

Setting $p=1$ results in the approximate solution of eqn. (A.1):

$$u=\lim_{p\to 1} v=v_0+v_1+v_2+... \quad (A.9)$$

This is the basic idea of the HPM.

Appendix: B

Analytical Expression of the Non-Linear Boundary Value Problem Eqns. (6)-(8) using the Homotopy Perturbation Method:

$$\frac{d^2W}{dy^2}+6\gamma\frac{d^2W}{dy^2}\left(\frac{dW}{dy}\right)^2=-1 \quad (B.1)$$

$$\frac{d^2\theta}{dy^2}+\lambda\left[e^{\left(\frac{\theta}{1+\varepsilon\theta}\right)}+m\left(\frac{dW}{dy}\right)^2\left(1+2\gamma\left(\frac{dW}{dy}\right)^2\right)\right]=0 \quad (B.2)$$

When $\varepsilon\theta$ is small, then the eqn. (B.2) can be written as

$$\frac{d^2\theta}{dy^2}+\lambda\left[(1+\theta-\varepsilon\theta^2)+m\left(\frac{dW}{dy}\right)^2+2m\gamma\left(\frac{dW}{dy}\right)^4\right]=0 \quad (B.3)$$

We construct the Homotopy for the eqns. (B.1) and (B.3) are as follows:

$$(1-p)\left[\frac{d^2W}{dy^2}+1\right]=hp\left[\frac{d^2W}{dy^2}+6\gamma\frac{d^2W}{dy^2}\left(\frac{dW}{dy}\right)^2+1\right] \quad (B.4)$$

$$(1-p)\left(\frac{d^2\theta}{dy^2}+\lambda\right)+p\left[\frac{d^2\theta}{dy^2}+\lambda\left((1+\theta-\varepsilon\theta^2)+m\left(\frac{dW}{dy}\right)^2+2m\gamma\left(\frac{dW}{dy}\right)^4\right)\right]=0 \quad (B.5)$$

The approximate solution of the eqns. (3.4.2.3) and (3.4.2.4) are as follows:

$$W=W_0+pW_1+p^2W_2+..... \quad (B.6)$$

$$\theta=\theta_0+p\theta_1+p^2\theta_2+..... \quad (B.7)$$

The initial approximations are as follows:

$$W_0'(0)=0\,;\,W_0(1)=0,\;\; \theta_0'(0)=0;\,\theta_0(1)=0, \quad (B.8)$$

$$W_i'(0)=0\,;\,W_i(1)=0,\;\; \theta_i'(0)=0;\,\theta_i(1)=0,\; i=1,2,3,... \quad (B.9)$$

Substituting the eqns. (B.6) and (B.7) into the eqns. (B.4) and (B.5) we get

$$(1-p)\left[\frac{d^2(W_0+pW_1+p^2W_2+...)}{dy^2}+1\right]+p\left[\begin{array}{l}\frac{d^2(W_0+pW_1+p^2W_2+...)}{dy^2}\\ +6\gamma\left(\frac{d^2(W_0+pW_1+p^2W_2+...)}{dy^2}\right)\\ \left(\frac{d(W_0+pW_1+p^2W_2+...)}{dy}\right)^2+1\end{array}\right]=0 \quad (B.10)$$

ISSN: 2456 – 3080
International Journal of Applied and Advanced Scientific Research
Impact Factor 5.255, Special Issue, February - 2017
International Conference on Advances in Theoretical and Applied Mathematics – ICATAM 2017
On 14th February 2017 Organized By
Madurai Sivakasi Nadars Pioneer Meenakshi Women's College, Poovanthi, Tamilnadu

$$(1-p)\left(\frac{d^2(\theta_0+p\theta_1+p^2\theta_2+....)}{dy^2}+\lambda\right)+p\begin{bmatrix}\frac{d^2(\theta_0+p\theta_1+p^2\theta_2+....)}{dy^2}\\ +\lambda\begin{pmatrix}1+(\theta_0+p\theta_1+p^2\theta_2+....)\\ -\varepsilon(\theta_0+p\theta_1+p^2\theta_2+....)^2\end{pmatrix}\\ +m\left(\frac{d(W_0+pW_1+p^2W_2+....)}{dy}\right)^2\\ +2m\gamma\left(\frac{d(W_0+pW_1+p^2W_2+....)}{dy}\right)^4\end{bmatrix}=0 \quad \text{(B.11)}$$

Comparing the coefficients of like powers p in the eqns. (B.10) and (B.11) we get

$$p^0:\frac{d^2W}{dy^2}+1=0 \quad \text{(B.12)}$$

$$p^0:\frac{d^2\theta}{dy^2}+\lambda=0 \quad \text{(B.13)}$$

$$p^1:\frac{d^2W_1}{dy^2}+\left[\frac{d^2W_0}{dy^2}+6\gamma\left(\frac{d^2W_0}{dy^2}\right)\left(\frac{dW}{dy}\right)^2+1\right]=0 \quad \text{(B.14)}$$

$$p^1:\frac{d^2\theta_1}{dy^2}+\left[\lambda(\theta_0-\varepsilon\theta_0^2)+m\lambda\left(\frac{dW_0}{dy}\right)^2+2m\gamma\lambda\left(\frac{dW_0}{dy}\right)^4\right]=0 \quad \text{(B.15)}$$

Solving the eqns. (B.12), (B.13), (B.14), (B.15) and using the boundary conditions (B.8) and (B.9) we can obtain the following results:

$$W_0(y)=\frac{(1-y^2)}{2} \quad \text{(B.16)}$$

$$W_1(y)=\frac{\gamma}{2}(1-y^4) \quad \text{(B.17)}$$

$$\theta_0(y)=\frac{\lambda}{2}(1-y^2) \quad \text{(B.18)}$$

$$\begin{aligned}\theta_1(y)=&\left[-\frac{\lambda^2}{2}+\frac{\lambda^3\in}{4}\right]\left(\frac{y^2}{2}\right)+\left[\frac{\lambda^2}{2}-\frac{\lambda^3\in}{2}-\lambda m\right]\left(\frac{y^4}{12}\right)+\left[\frac{\lambda^3\in}{4}-4\lambda m\gamma-2\lambda\gamma m\right]\left(\frac{y^6}{30}\right)\\ &+\left[-4\lambda m\gamma^2-16\gamma^2\lambda m\right]\left(\frac{y^8}{56}\right)-\left(\frac{48}{90}\right)\left(\lambda m\gamma^3y^{10}\right)-\left(\frac{64}{132}\right)\left(\lambda m\gamma^4y^{12}\right)\\ &-\left(\frac{32}{182}\right)\left(\lambda\gamma^5y^{14}\right)+B\end{aligned} \quad \text{(B.19)}$$

According to the HPM, we conclude that

$$W=\lim_{p\to1}W=W_0+W_1+... \quad \text{(B.20)}$$

$$\theta=\lim_{p\to1}\theta=\theta_0+\theta_1+... \quad \text{(B.21)}$$

After putting the eqns. (B.16) and (B.17) into the eqn. (B.20) and (B.18) and (B.19) into the eqn. (B.21) we get the solutions in the text eqns. (11) - (13).

Appendix: C

Nomenclature:

Symbol	Meaning
y	Dimensionless distance
W	Dimensionless velocity
θ	Dimensionless temperature
T	Absolute Temperature
U	Fluid characteristic velocity

T_0	Plate temperature
k	Thermal conductivity of the material
Q	Heat of reaction
A	Rate Constant
E	Activation Energy
R	Universal Gas Constant
C_0	The initial concentration of the reactant species
a	Channel Half Width
β_3	Material Coefficient
P	Modified Pressure
μ	Fluid dynamic viscosity coefficient
λ	Frank- Kamenetskii parameter
$\in$	Action energy parameter
γ	Dimensionless non-Newtonian parameter
m	Dimensionless viscous heating parameter
C_f	Skin- Friction Parameter
Nu	Nusselt Number
l_w	Shear Stress
q_w	Heat flux evaluated at the wall
$U(\lambda)$	Local representation of an algebraic function of λ
H	Constant
λ_c	Critical Point
α	Exponent